MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXIII.

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WITH MANY

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IN ADDITION TO THOSE

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AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

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CONTENTS.

Solved Questions.

CORRIGENDA.

Vol. 63, p. 52, line 6 from bottom, in numerator, *read* S for *s*; line 2 from bottom, *for* $\frac{1}{2}$ ( ) *read* $\frac{1}{2}$ (C′M − A′L); and in last line, *read* "bisects A′C′."

MATHEMATICS

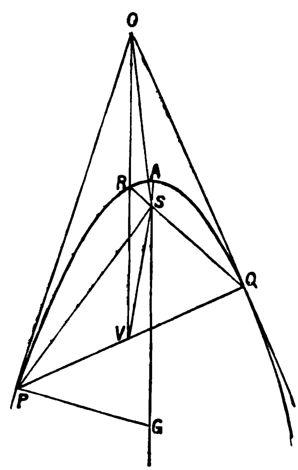
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3424. (Professor GENESE, M.A.)—Prove that the product of the two normals that can be drawn to a parabola from a point on it is 2SP . PG.

Solution by H. W. CURJEL, M.A.; Prof. KRISHNACHANDRA; *and others.*



Let the normals at Q, Q′ cut the parabcla again in P; let V be the middle point of PQ, and RV a diameter; then RQ passes through S. Let O be the pole of PQ; then we have

$$QV^2 = 4SR \,.\, RV = 4SR \,.\, RQ,$$

$$SQ \,.\, SP + SV^2 = SO^2 + SV^2$$
$$= 2SR^2 + 2RV^2$$
$$= 2SR^2 + 2RQ^2,$$

$$SQ^2 + SP^2 = 2SV^2 + 2QV^2$$
$$= 2SV^2 + 8SR \,.\, RQ;$$

$$\therefore \quad SP^2 + 2SQ \,.\, SP + SQ^2$$
$$= 4\,(SR^2 + RQ^2 + 2SR \,.\, RQ);$$

$$\therefore SP = 2SR + 2RQ - SQ = 4SR + SQ.$$

But $$SR = \frac{AS \,.\, SQ}{SQ - SA};$$

$$\therefore \quad SP = \frac{3AS \,.\, SQ + SQ^2}{SQ - AS};$$

$$\therefore SQ^2 - SQ\,(SP - 3AS) + AS \,.\, SP = 0.$$

Hence $$SQ \,.\, SQ' = AS \,.\, SP, \quad SQ + SQ' = SP - 3AS.$$

But $$QP^2 = 4QV^2 = 16SR \,.\, RQ = \frac{16AS \,.\, SQ^3}{(SQ - AS)^2};$$

therefore $$QP \,.\, Q'P = \frac{16 \,.\, SQ \,.\, SQ' \,.\, AS\,(SQ \,.\, SQ')^{\frac{1}{2}}}{SQ \,.\, SQ' + AS^2 - AS\,(SQ + SQ')}$$
$$= 2SP\,(4SA \,.\, SP)^{\frac{1}{2}} = 2SP \,.\, PG.$$

[Professor LAMPE gives his solution as follows: — Let $y^2 = 2px$

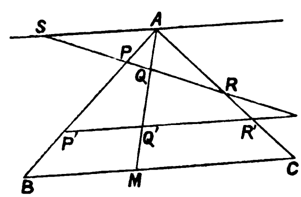
be the equation of the parabola, $p(y-y_1) = -y_1(x-x_1)$ the equation of the normal in point (x_1, y_1). Suppose (x, y) to be a point given on the parabola, (x_1, y_1) its foot; then, substituting $x = y^2/2px$, $x_1 = y_1^2/2px$, we get $y_1^2 + y_1 y + 2p^2 = 0$, an equation whose roots, y_1 and y_2, satisfy the relations $y_1 + y_2 = -y$, $y_1 y_2 = 2p^2$. The lengths of the normals N_1 and N_2 between (x, y), (x_1, y_1), and (x, y), (x_2, y_2) result from

$$N_1^2 = (y-y_1)^2 + (x-x_1)^2 = (y-y_1)^2 + (y^2-y_1^2)^2/4p^2$$
$$= (y-y_1)^2\{1 + (y+y_1)^2/4p^2\},$$
$$N_2^2 = (y-y_2)^2\{1 + (y+y_2)^2/4p^2\}.$$

Multiplying and reducing by aid of $y_1 + y_2 = -y$, $y_1 y_2 = 2p^2$, we arrive at $N_1 . N_2 = (y^2+p^2)^{\frac{3}{2}}/p$. Remembering that $(y^2+p^2)^{\frac{3}{2}} = n$, and $n^3/p^2 = R$, if R is the radius of curvature at (x, y), the result changes to $N_1 . N_2 = R . p$, and this seems to be the meaning of 2SP.PG in the enunciation of the question, which is not quite clear.]

12525. (C. E. Hillyer, M.A.)—If ABC be a triangle, M the middle point of BC, and through any point O a straight line be drawn meeting AB in P, AM in Q, AC in P, and a parallel to BC through A in S, show that OP/PS + OR/RS = 2OQ/QS, and hence deduce all the properties of a harmonic range.

Solution by Rev. D. T. Griffiths, B.A.; Prof. Chakrivarti; *and others.*



Let OR′Q′P′ be the parallel to BC through O; then

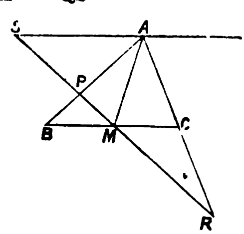
$$\frac{OP}{SP} = \frac{OP'}{SA},$$

because triangles SAP, OP′P are similar.

Again, $$\frac{OR}{SR} = \frac{OR'}{SA},$$

because triangles SAR, OR′R are similar; adding,

$$\frac{OP}{SP} + \frac{OR}{SR} = \frac{OP' + OR'}{SA} = 2\,\frac{OQ'}{SA} = 2\,\frac{OQ}{QS}.$$



Let O coincide with M; therefore

$$\frac{MP}{SP} + \frac{MR}{SR} = 0,$$

therefore $$\frac{MR}{MP} = -\frac{SR}{SP};$$

therefore (RMPS) = − 1.

This proves Prop. 5, Section iii., Book vi., Casey's "Sequel."

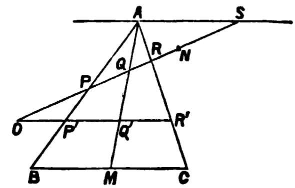
[The PROPOSER gives the following solution:—

(1) Let O coincide with P; then $\frac{PR}{RS} = 2\frac{PQ}{QS}$,

$\therefore$ PR.QS = 2PQ.RS.

(2) Let O coincide with Q; then

$\frac{QP}{PS} + \frac{QR}{RS} = 0, \quad \therefore \frac{QP}{QR} = -\frac{SP}{SR}$.



(3) It follows from (1) and (2) that RP.QS = 2RQ.PS.

(4) Let O′ be another point in OS;

then $\quad \frac{O'P}{PS} + \frac{O'R}{RS} = 2\frac{O'Q}{QS}$;

$\therefore$ by subtraction, $\quad \frac{OO'}{PS} + \frac{OO'}{RS} = 2\frac{OO'}{QS}; \quad \therefore \frac{1}{PS} + \frac{1}{RS} = \frac{2}{QS}$.

Also, if O coincide with N, the mid-point of QS, we have

$$\frac{NP}{PS} + \frac{NR}{RS} = 2\frac{NQ}{QS} = 1.$$

$$\therefore \quad \frac{NR}{RS} = 1 - \frac{NP}{PS} = \frac{NS}{PS} = \frac{NQ}{PS}; \quad \therefore \frac{NR}{NQ} = \frac{RS}{PS} = \frac{QR}{PQ}.$$

Also $\quad \frac{NP}{PS} = 1 - \frac{NR}{RS} = \frac{NS}{RS} = \frac{NQ}{RS}; \quad \therefore \frac{NQ}{NP} = \frac{RS}{PS} = \frac{QR}{PQ}$.

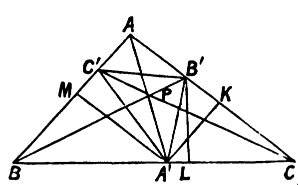
Thus (5) $\quad NP.NR = NQ^2$,

and (6) $\quad \frac{NR}{NP} = \frac{NQ^2}{NP^2} = \frac{QR^2}{PQ^2}$.]

10222. (Professor DÉPREZ.)—Soient P un point du plan du triangle ABC, et A′, B′, C′ les points où les droites AP, BP, CP rencontrent les côtés BC, CA, AB. Le lieu d'un point P tel que l'angle de Brocard du triangle A′B′C′ ait une valeur donnée V, est une courbe du sixième ordre.

Solution by H. J. WOODALL, A.R.C.S.

$\cot w = \cot A + \cot B + \cot C$
$= R(a^2 + b^2 + c^2)/abc \;\ldots\ldots\; (1)$



(MILNE, *Companion*, p. 107). Whence we conclude that cot w is always positive, and hence $w \not> \frac{1}{2}\pi$.

Also $4\cos^2 w\,(b_1^2c_1^2 + c_1^2a_1^2 + a_1^2b_1^2)$
$= (a_1^2 + b_1^2 + c_1^2)^2 \;\ldots\ldots\; (2)$.

If the bracket on the left-hand side has in its new form an algebraical square root, we may, hence, take the positive value of that root and get

$$2\cos w\,(b_1^2c_1^2+c_1^2a_1^2+a_1^2b_1^2)^{\frac{1}{2}} = a_1^2+b_1^2+c_1^2.$$

If then w has the given value V, this relation will be a relation between the coordinates of P (a_1, β_1, γ_1).

By ordinary work we find

$$c_1^2\sin^2 C\,(a\alpha_1+c\gamma_1)^2\,(b\beta_1+c\gamma_1)^2$$
$$= 4c^2\Delta^2\left\{\alpha_1^2\beta_1^2+\beta_1^2\gamma_1^2+\gamma_1^2\alpha_1^2+2\alpha_1\beta_1\gamma_1\,(\alpha_1\cos A+\beta_1\cos B-\gamma_1\cos C)\right\},$$

and similar equations for a_1 and b_1. Substitution of these will make (2) an equation of the twelfth order. Its algebraical square root (if such can be found) would be of the sixth order, as required.

1031. (Editor.)—A solid is formed by the revolution of the Cissoid of Diocles about the diameter $(=a)$ of the generating circle. Show that the attraction of the solid upon a particle at its vertex is equal to that of a sphere, whose diameter is a, upon a particle at its surface.

Solution by Professor Lampe, LL.D.

After Thomson and Tait ("Natural Philosophy," § 477), the attraction of a circular disc upon a point in the axis of the disc is found to be

$$A = 2\pi\rho\left(1-\frac{h}{(h^2+a^2)^{\frac{1}{2}}}\right),$$

a being the radius of the disc. Therefore, the equation of the cissoid being $y^2 = x^3/(a-x)$, the attraction of an infinitesimal disc, perpendicular to the axis of x, will be

$$dA = 2\pi\rho\left(1-\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)dx,$$

$$A = 2\pi\rho\int_0^a\left\{1-\left(\frac{a-x}{a}\right)^{\frac{1}{2}}\right\} = 2\pi\rho\,(a-\tfrac{2}{3}a) = \tfrac{2}{3}\pi\rho a.$$

This is at the same time the attraction of a sphere, of density ρ, upon a particle, density equal to unity, at its surface.

But, if we calculate the volume of this solid

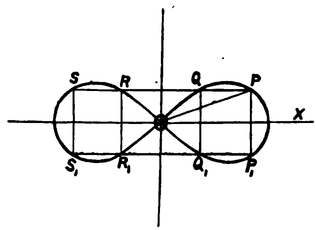
$$v = \pi\int_0^a y^2dx = \pi\int_0^a\left(\frac{a^3}{a-x}-a^2-ax-x^2\right)dx^2$$

$$= \pi\left[a^2l\frac{a}{a-x}-a^2x-\tfrac{1}{2}ax^2-\tfrac{1}{3}x^3\right]_{x=a},$$

this proves to be *infinite;* whence the solid ought not to be considered as a *body* whose idea involves the notion of a finite mass.

3318. (R. Tucker, M.A.)—Inscribe the maximum rectangle in a lemniscate.

Solution by Professor Lampe, LL.D.



1. A rectangle such as PSS_1P_1, having its vertices equidistant from the centre O.

From the equation $r^2 = a^2 \cos 2\theta$, we get $\cos 2\theta_1 = \cos 2\theta_2$, if θ_1 and θ_2 are the angles corresponding to the vertices P and S.

$$PP_1 = 2a\,(\cos 2\theta)^{\frac{1}{2}} \sin \theta,$$
$$PS = 2a\,(\cos 2\theta)^{\frac{1}{2}} \cos \theta,$$

or area of rectangle $= 4a^2 \cos 2\theta \sin \theta \cos \theta = 2a^2 \sin 2\theta \cos 2\theta = a^2 \sin 4\theta$.

This becomes a maximum for $\sin 4\theta = 1$; whence $4\theta = 90^\circ$, $\theta = 22\frac{1}{2}^\circ$.

2. A rectangle, such as PQQ_1P_1, having its vertices not equidistant from the centre O.

Putting angle $POX = \theta_1$, $PO = r_1$, $QOX = \theta_2$, $QO = r_2$, we have $r_1 \sin \theta_1 = r_2 \sin \theta_2$, but not $\theta_1 = \theta_2$. Squaring and substituting, we get

$$a^2 \cos 2\theta_1 \sin^2 \theta_1 = a^2 \cos 2\theta_2 \sin^2 \theta_2,$$

or $$\cos 2\theta_1\,(1-\cos 2\theta_1) = \cos 2\theta_2\,(1-\cos 2\theta_2);$$

hence $$\cos 2\theta_1 - \cos 2\theta_2 = \cos^2 2\theta_1 - \cos^2 2\theta_2,$$

and $$1 = \cos 2\theta_1 + \cos 2\theta_2 \qquad\qquad (1).$$

The area of PQQ_1P_1 will be

$$A = r_1^2 \sin 2\theta_1 - r_2^2 \sin 2\theta_2 \quad (\text{because } r_1 \sin\theta_1 = r_2 \sin\theta_2)$$
$$= \tfrac{1}{2}a^2\,(\sin 4\theta_1 - \sin 4\theta_2).$$

Differentiating, $dA/d\theta_1 = 2a^2\,(\cos 4\theta_1 - \cos 4\theta_2 \,.\, d\theta_2/d\theta_1)$.

But, from (1), follows $d\theta_2/d\theta_1 = -\sin 2\theta_1/\sin 2\theta_2$.

Making then $dA/d\theta_1 = 0$, we find

$$\cos^2 4\theta_1 \sin^2 2\theta_2 = \cos^2 4\theta_2 \sin^2 2\theta_1.$$

Now $\cos 4\theta = 2\cos^2 2\theta - 1$, $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$;

reducing by these formulas, we get

$$\cos^2 2\theta_1 + \cos^2 2\theta_2 = \cos^2 2\theta_1 \,.\, \cos^2 2\theta_2 + \tfrac{3}{4} \qquad\qquad (2).$$

The equations (1) and (2) lead to the values of $\cos 2\theta_1$ and $\cos 2\theta_2$, viz.,

$$\cos 2\theta_1 = \tfrac{1}{2}\left\{1 + (5-\sqrt{20})^{\frac{1}{2}}\right\}, \quad \cos 2\theta_2 = \tfrac{1}{2}\left\{1 - (5-\sqrt{20})^{\frac{1}{2}}\right\}.$$

The solution of Case 2 may be made in rectangular coordinates in the following manner. Solving the equation for x, we have

$$x^2 = \tfrac{1}{2}a^2 - y^2 \pm (\tfrac{1}{4}a^4 - 2a^2y^2)^{\frac{1}{2}}.$$

The area A of PQQ_1P_1 will be $A = 2y\,(x_1 - x_2)$,

$$A^2 = 4y^2\,(x_1^2 + x_2^2 - 2x_1r_2) = 4y^2\left\{a^2 - 2y^2 - 2y\,(a^2+y^2)^{\frac{1}{2}}\right\}.$$

Putting $d\,(A^2) = 0$, we find $y^4 + a^2y^2 - \frac{1}{16}a^4 = 0$, whence

$$y^2 = \tfrac{1}{4}a^2\,(-2+\sqrt{5}), \quad A = a^2\left\{\tfrac{5}{2}\sqrt{5} - \tfrac{11}{2}\right\}^{\frac{1}{2}} = 0\cdot30028a^2.$$

[Mr. Curjel puts the solution thus:—The equation to the lemniscate is

$$x^4 + 2x^2y^2 + y^4 + 2a^2(y^2 - x^2) = 0.$$

Area of rectangle $= 4u = 4xy$. For a critical value

$$xdy + ydx = du = 0 \quad \text{and} \quad (x^3 + xy^2 - a^2x)\,dx + (y^3 + yx^2 + a^2y)\,dy = 0\,;$$

therefore x is given by $(2x^2 - a^2)(2x^2 + a^2) - 4a^2x^2 = 0$; therefore

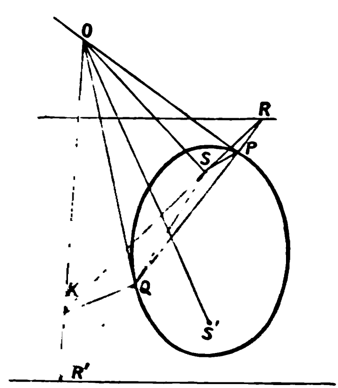
$$x = \tfrac{1}{2}a\,(2 + 2\sqrt{2})^{\frac{1}{2}}, \quad y = \tfrac{1}{2}a\,(2\sqrt{2} - 2)^{\frac{1}{2}},$$

and these values make u a maximum.

Prof. Lampe remarks that the solution by Mr. Curjel refers only to rectangles PP_1S_1S, and neglects rectangles PQQ_1P_1 or PP_1R_1R.]

3160. (Rev. C. Taylor, D.D.)—If a normal meet the conic again in Q, and the directrices in R, R′, then (O being the pole of the chord, and S, S′ the foci) SR, OR′ and S R′, OR intersect on the normal at Q.

Solution by H. W. Curjel, M.A.



Let RS, OR′ meet in K. Now OS, SR are the internal and external bisectors of the ∠ PSQ.

$\therefore$ ∠ RSO = a right angle
= RPO;

therefore RPSO are concyclic;

$\therefore$ ∠ POR = PSR = QSK.

Again OR′, OS′ and OR, OS are harmonic conjugates with respect to OP, OQ, and

∠ S′OQ = SOP;

$\therefore$ ∠ R′OQ = ROP = KSQ;

therefore OSQK are concyclic;

$\therefore$ ∠ OQK = OSK
= right angle,

i.e., RS, OR′ cut on the normal at Q. Similarly, R S′, OR cut on the normal at Q.

3390. (Professor Evans, M.A.)—Find the area of the maximum ellipse that can be inscribed in the quadrant of a given circle.

Solution by Professor Lampe, LL.D.

1. See Fig. 1. The centre M of the circle is a point of the director-circle of the ellipse, whence $MC = (a^2+b^2)^{\frac{1}{2}}$, a and b being the semi-axes. $MB = MC + CB, = a + (a^2+b^2)^{\frac{1}{2}} = r$, or $(r-a)^2 = a^2+b^2$, $r^2 - 2ra = b^2$. The area $A = ab\pi = b\pi(r^2-b^2)/2r$. Forming $dA/db = 0$, we get

$$b^2 = \tfrac{1}{3}r^2, \quad a = \tfrac{1}{3}r, \quad A = \tfrac{1}{9}r^2\sqrt{3}\,\pi.$$

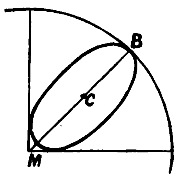


Fig. 1.

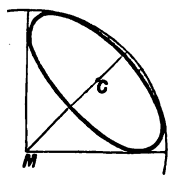


Fig. 2.

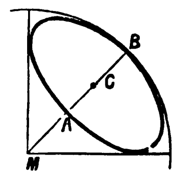


Fig. 3.

2. See Fig. 2. Putting $MC = \alpha$ and taking MC as axis of x, the equation of the ellipse will be $(x-\alpha)^2/a^2 + y^2/b^2 = 1$, that of the circle $x^2+y^2 = r^2$, and moreover $\alpha^2 = a^2+b^2$. Eliminating y^2 and forming the condition for equal roots of the resulting quadric, we get

$$ab^4 = r^2(b^2-a^2), \quad a^2 = b^2 - 2b^4/r^2.$$

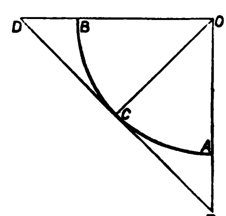
Substituting in $\qquad A^2 = a^2b^2\pi^2 = \pi^2\left\{b^4 - 2b^6/r^2\right\},$

we find for $dA/db = 0$ the same values as in 1, $b^2 = \frac{1}{3}r^2$, $a = \frac{1}{3}r$.

This gives (see Fig. 3) $a = AC = CB = MA = \frac{1}{3}r$. The radius of curvature of the ellipse in point B is $b^2/a = 3a = r$, or the given circle is the circle of curvature for the ellipse with maximum of area.

[Mr. Curjel gives his solution as follows:—

Let OACB be the quadrant, O being the centre, and C the middle point of arc ACB. Draw tangent at C meeting OA, OB in E and D. The maximum ellipse in ΔOED touches DE at C, and its radius of curvature at C = OC; therefore it lies entirely within the quadrant. Hence it is also the maximum ellipse inscribed in the quadrant, and its area evidently



$$= \frac{\pi OC^2}{3\sqrt{3}}.]$$

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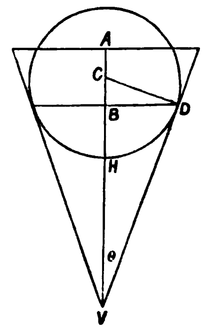
863. (Editor.)—If a conical glass, whose altitude is a and the generating angle θ, be filled with water, find the radius of the sphere

which, being put into it, shall cause the greatest quantity of water to overflow.

Solution by Professor LAMPE, LL.D.



Let $CV = u$,

$CD = r = u \sin\theta$,

$AH = h = a - u + u\sin\theta$.

The volume of water dislodged by the sphere is a spherical segment,

$$= \tfrac{1}{3}\pi h^2 (3r - h),$$

whose differential, expressed in u, becomes

$$\tfrac{1}{3}\pi \left\{a - u(1-\sin\theta)\right\} \times \left\{3a - 3u(1-\sin\theta)(2\sin\theta+1)\right\} du;$$

$$\therefore\ u = \frac{a}{(1-\sin\theta)(2\sin\theta+1)}, \qquad r = u\sin\theta.$$

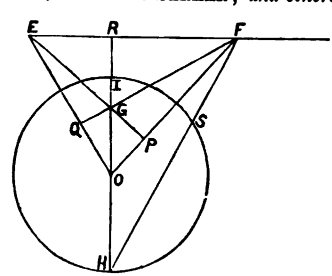
Calculating $\qquad h = \dfrac{2a\sin\theta}{2\sin\theta+1}$,

and $\qquad AB = a - u + r\sin\theta = \dfrac{a\sin\theta}{2\sin\theta+1}$,

we see that the plane of the circle of contact bisects in B the height h of the spherical segment which represents the maximum. [A solution by Professors ZERR and MUKHOPADHYAY is given on page 106 of Vol. LXI.]

12511. (Professor MOREL.)—Étant donné un quadrilatère inscriptible ABCD, on mène le diamètre HI perpendiculaire à la troisième diagonale EF; soit KC symétrique par rapport à EF de l'extrémité I la plus rapprochée de EF. Démontrer que les quatre points E, F, H, K sont sur une même circonférence.

Solution by H. W. CURJEL, M.A.; Professor SARKAR; *and others.*



Let G be the intersection of AC, BD. Then the ΔEFG is self-conjugate; therefore the centre O of the circle is its orthocentre. Let HF meet the circle in S, and let EG, FO; EO, FG; OG, EF meet in P, Q, R. Then

$$FE \,.\, FR = FG \,.\, FQ$$
$$= FS \,.\, FH,$$

since EF is the radical axis of circle ISH, and circle on GO as diameter; therefore S lies

on circle through ERH ; $\angle$ HSE = a right angle = HSI; therefore EI is perpendicular to FH ; therefore I is the orthocentre of $\triangle$EFH ; therefore E, F, H, K are concyclic.

9666. (Professor DE LONGCHAMPS.)—Trouver le lieu des centres des coniques qui ont un sommet donné et qui passent par deux points donnés.

Solution by H. J. WOODALL, A.R.C.S.

Let (a, b) be the given vertex, (h, k), (h_1, k_1) the other given points; let (X, Y) be the centre, e the eccentricity,

$$x \cos\alpha + y \sin\alpha - p = 0, \text{ the directrix};$$

$$\therefore \quad 1-e = (a\cos\alpha + b\sin\alpha - p) / (X\cos\alpha + Y\sin\alpha - p);$$

$$\therefore \quad e = \{(X-a)\cos\alpha + (Y-b)\sin\alpha\} / \{X\cos\alpha + Y\sin\alpha - p\}.$$

The focus is given by $x - X = e(a-X)$, $y-Y = e(b-Y)$; therefore, since a conic is the locus of points whose distances from a given point are in a given ratio (e) to their distances from a given line, we have three equations like

$$\{X + e(a-X) - a\}^2 + \{Y + e(b-Y) - b\}^2 = e^2 (a\cos\alpha + y\sin\alpha - p)^2;$$

also, since (a, b) is the vertex, $(Y-b) = (X-a)\tan\alpha$. We have thus five equations (equivalent to four) to eliminate e, α, p. We may make (a, b) the origin. Our four equations become

$$e = (X\cos\alpha + Y\sin\alpha) / (X\cos\alpha + Y\sin\alpha - p) \quad \text{.............(1)},$$

$$(X-h-eX)^2 + (Y-k-eY)^2 = e^2(h\cos\alpha + k\sin\alpha - p)^2 \quad \text{.........(2)},$$

$$(X-h_1-eX)^2 + (Y-k_1-eY)^2 = e^2(h_1\cos\alpha + k_1\sin\alpha - p)^2 \quad \text{......(3)},$$

$$Y = X\tan\alpha \quad \text{..................................(4)},$$

and elimination gives us the sextic.

$$8(X^2+Y^2)\{(h^2+k^2)(h_1Y-k_1X)^2 - (h_1^2+k_1^2)(hY+kX)^2\}$$
$$\times\{(h^2+k^2)(h_1X+k_1Y) - (h_1^2+k_1^2)(hX+kY)\}$$
$$= [(hY-kX)^2(h_1^2+k_1^2-2h_1X-2k_1Y)$$
$$-(h_1Y-k_1X)^2(h^2+k^2-2hX-2kY)]^2.$$

12303. (Professor DE ROCQUIGNY.)—Posons $T_a = \frac{1}{2}[a(a+1)]$, a étant un nombre entier quelconque, positif ou négatif. Démontrer que le produit de la somme de trois triangulaires $T_a + T_b + T_c$, par la somme de trois triangulaires $T_m + T_n + T_p$, est une somme de trois triangulaires lorsque $a+b+c=0$, $m+n+p=0$.

Solution by H. W. CURJEL, B.A.; Professor CHAKRIVARTI; *and others.*

Let $a, b, c, m, n, p = y-z,\ z-x,\ x-y,\ \eta-\zeta,\ \zeta-\xi,\ \xi-\eta$ respectively.

Then $$(T_a + T^b + T_c)(T_m + T_n + T_p)$$
$$= (x^2 + y^2 + z^2 - yz - zx - xy)(\xi^2 + \eta^2 + \zeta^2 - \eta\zeta - \zeta\xi - \xi\eta)$$
$$= X^2 + Y^2 + Z^2 - YZ - ZX - XY = T_{Y-Z} + T_{Z-X} + T_{X-Y},$$
where $X = x\xi + z\eta + y\zeta, \quad Y = z\xi + y\eta + x\zeta, \quad Z = y\xi + x\eta + z\zeta.$

12246. (R. Chartres.)—Express $(\cdot\dot{0}00 \dots \dot{1})^2$ in any scale as a recurring fraction, stating the law of the sequence of the figures, and explaining the exception. Of what finite series is the period, considered as whole numbers, the sum?

Solution by R. Chartres; Professor Bhattacharya; *and others.*

For simplicity of notation take $(\cdot\dot{0}0\dot{1})^2$ in the denary scale

$$= \left(\frac{1}{1000}\right)^2 \left(1 - \frac{1}{1000}\right)^{-2} = \cdot 000\ 001\ 002 \dots (998\text{C})(999\text{B})(1000\text{A})\dots$$

$$= \cdot\dot{0}00\ 001\ 002 \dots 997\ 99\dot{9},$$

since A makes B 000, and C 999 leaving a gap between 997 and 999.

Thus $\quad (\cdot\dot{1})^2 = \cdot\dot{0}1234567\dot{9}, \quad (\cdot\dot{0}\dot{1})^2 = \cdot\dot{0}0\ 01\ 02 \dots 979\dot{9}.$

Let P be the full period of $n\,(r^n - 1)$ recurring figures, r = radix;

therefore $$\left(\frac{1}{r^n - 1}\right)^2 = \frac{P}{r^{n(r^n-1)} - 1};$$

therefore $$P = \frac{(1+p)^p - 1}{p^2}, \quad \text{if} \quad r^n - 1 = p;$$

therefore $$P = 1 + C_2 + pC_3 + p^2C_4 \dots + p^{p-2},$$

where C_q = combinations of p, or $(r^n - 1)$ things q together. This can be easily verified by taking a low radix.

12377. (Professor Morley, M.A.)—An epitrochoid rotates in its plane about its centre; prove that the locus of points of contact of tangents in a given direction is a circular quartic.

Solution by H. W. Curjel, B.A.; Professor Wachter; *and others.*

The equation to any of the epitrochoids may be written

$$x = (a+b)\cos(\theta + \alpha) - d\cos\left(\frac{a+b}{b}\theta + \alpha\right) \dots\dots\dots\dots\dots(1),$$

$$y = (a+b)\sin(\theta + \alpha) - d\sin\left(\frac{a+b}{b}\theta + \alpha\right) \dots\dots\dots\dots\dots(\ \ ),$$

where a is different for different curves. The fixed direction may be taken parallel to the axis of y without loss of generality. Hence the locus will be given by eliminating θ and a between (1), (2), and $dy/d\theta = 0$, *i.e.*, between

$$x = (a+b)\cos\phi - d\cos\psi,$$
$$y = (a+b)\sin\phi - d\sin\psi, \quad b\cos\phi = d\cos\psi;$$

where $$\phi = \theta + a, \quad \psi = \frac{a+b}{b}\theta + a.$$

Then $x = a\cos\phi$, and locus is

$$(x^2+y^2)\left\{x^2(a+2b)^2 + a^2y^2\right\} - 2a^2y^2\left\{(a+b)^2+d^2\right\}$$
$$-2x^2\left[\left\{(a+b)^2+d^2\right\}^2 - (b^2+d^2)\left\{(a+b)^2-d^2\right\} + b^2d^2\right]$$
$$+a^2\left\{(a+b)^2-d^2\right\}^2 = 0,$$

a circular quartic.

12426. (H. J. Woodall, A.R.C.S.) — Prove that $(a^2+ab+b^2)\times(x^2+xy+y^2)$ can be put into the form X^2+XY+Y^2.

Solution by R. J. Dallas; H. Fortey; *and others.*

The following theorem includes the theorem of the question as a particular case. Take $n-1$ letters $abc\ldots jk$, and form a circulant of n rows with zeros in the leading diagonal. If we remove the factor $a+b+c+\ldots+j+k$, we get a determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ k & 0 & a & b & \ldots & j \\ \ldots & & \ldots & & \ldots & \end{vmatrix}, \text{ which we call } F(abc\ldots jk).$$

We have then to prove that the product of $F(abc\ldots jk)$ and $F(xyz\ldots st)$ is a function of the same form.

Multiplying the circulants whose first rows are

$$0\ abc\ldots jk, \quad \text{and} \quad 0\ xyz\ldots st, \text{ respectively,}$$

and removing from the resulting circulant the factor

$$(a+b+c+\ldots+j+k)(x+y+z+\ldots+s+t),$$

we see that $$F(abc\ldots jk)\times F(xyz\ldots st),$$

is equal to the determinant

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ (ay+bz+\ldots+jt), & (ax+by+cz+\ldots), & (bx+cy+\ldots+ks), & (at+cx+\ldots)\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

Multiply the first row by $ax+by+cz+\ldots+kt$, and subtract it from each of the other rows.

We see thus that $F(abc\ldots jk)\times F(xyz\ldots st)$,

is $$F\left\{\begin{array}{l} -ax+(a-b)y+(b-c)z+\ldots+(j-k)t \\ (b-a)x+(c-b)y+(d-c)z+\ldots-kt \\ (c-a)x+(d-b)y+(e-c)z+\ldots-js+(a-k)t \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}.$$

In particular $F(a, b) = a^2 - ab + b^2$, $F(x, y) = x^2 - xy + y^2$;

$\therefore\quad F(a, b) \times F(x, y) = F\left[\{-ax + (a-b)y\},\ \{(b-a)x - by\}\right].$

Or $$(a^2 - ab + b^2)(x^2 + xy + y^2)$$
$$= \{-ax + (a-b)y\}^2 - \{-ax + (a-b)y\}\{(b-a)x - by\} + \{(b-a)x - by\}^2.$$

We have only to change the signs of b and y to get the product of $(a^2 + ab + b^2)$ and $(x^2 + xy + y^2)$.

12316. (The late W. J. C. Sharp, M.A.)—Prove that

$$\begin{vmatrix} 1+x_1, & 1, & 1, & \dots & 1, & 1, & 1, \\ 1, & 1+x_2, & 1, & \dots & 1, & 1, & 1, \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & 1, & 1, & \dots & 1+x_n, & 1, & 1, \\ 1, & 1, & 1, & \dots & 1, & 0, & 1, \\ 1, & 1, & 1, & \dots & 1, & 1, & 0, \end{vmatrix}$$

$$= x_1 x_2 \dots x_n \left\{ -1 + \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right\}.$$

Solution by H. W. Curjel, B.A; Professor Sarkar; *and others.*

The term of the highest degree in the x's is evidently $-\Pi x$; also the determinant is of the first degree in each of the x's, and is unaltered by interchanges of any of the x's.

Let u_{n-1} = value of the determinant when $x_n = 0$. Then it may easily be shown that $u_{n-1} = x_{n-1} u_{n-2} = x_{n-1} x_{n-2} u_{n-3}$, and so on, and

$$u_1 = \begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & 1+x_1, & 1, & 1 \\ 1, & 1, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = x_1; \qquad \therefore\quad u_{n-1} = \frac{\Pi x}{x_n}.$$

Hence the determinant

$$= -\Pi x + \Sigma\left(\frac{\Pi x}{x}\right) = x_1 x_2 x_3 \dots x_n \left\{ -1 + \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right\}.$$

12304. (Professor Mukhopadhyay.) — Prove that the number of ways in which p things may be distributed among q persons so that everybody may have one at least is

$$q^p - q(q-1)^p + \frac{q(q-1)}{2!}(q-2)^p - \dots.$$

Solution by H. W. CURJEL, B.A.; Professor BHATTACHARYA; *and others.*

The formula given is evident: since the first term gives the number of ways of distributing the n things when any number of persons may have none, and the second term gives the number of cases in which one person has none, a case in which r persons have none being counted r times, the third term gives the number of cases in which two persons have received none, a case in which r persons have none being counted $\frac{r(r-1)}{2}$ times, and so on. But

$$-r+\frac{r(r-1)}{2}-\frac{r(r-1)(r-2)}{3\,.\,2}+\&c. = (1-1)^r-1 = -1.$$

Hence formula is true.

2739. (Professor CROFTON, F.R.S.)—(1) Two points are taken at random within a circle, and a random straight line crosses the circle; find the probability that it shall pass between the points. (2) A straight line taken at random crosses any convex area; let p_1 be the probability that it passes between two points taken at random in the area; again, let p_2 be the probability that it meets the triangle formed by three points taken at random in the area; then $p_2 = \frac{3}{2}p_1$.

Solution by Professor ZERR.

Let M = the mean distance between the two points, L = the perimeter of the convex area; then

$$p_1 = 2\mathrm{M}/\mathrm{L} = 2\left(\frac{128r}{45\pi}\right)\Big/2\pi r = \frac{128}{45\pi^2} \text{ for the circle,}$$

$$p_2 = 3\mathrm{M}/\mathrm{L}; \qquad \therefore\; p_2 = \tfrac{3}{2}p_1.$$

8489. (MAURICE D'OCAGNE.)—Soient P_1 et P_2 deux paraboles de même foyer O et de même axe $\mathrm{O}x$. Par le point O menons une droite variable qui coupe P_1 en a_1, P_2 en a_2, et prenons sur cette droite le point a tel que $(\mathrm{O}a)^2 = \mathrm{O}a_1\,.\,\mathrm{O}a_2$. Le point a décrit une parabole P ayant aussi O pour foyer et $\mathrm{O}x$ pour axe. Cela posé, soient γ, γ_1 et γ_2 les centres de courbure des paraboles P, P_1 et P_2, répondant aux points a, a_1 et a_2. Démontrer que la droite $\mathrm{O}\gamma$ est conjuguée harmonique de la droite $\mathrm{O}a_1aa$ par rapport aux droites $\mathrm{O}\gamma_1$ et $\mathrm{O}\gamma_2$.

Solution by H. J. WOODALL, A.R.C.S.

The equation to a parabola under such circumstances is $l/r = 1+\cos\theta$. If $f(l_1l_2)$ be homogeneous of first degree and not necessarily integral and not involving θ, then $f(l_1l_2)/r = 1+\cos\theta$ is also a parabola. It is also easy to see that $\mathrm{O}\gamma_2\gamma\gamma_1$ is a straight line.

[The PROPOSER remarks that—"L'énoncé de Quest. 8489 ci-joint

provenait de la particularisation de certain résultat très général que j'avais obtenu il y a nombre d'années en étudiant l'inversion généralisée. Je n'avais pas pris garde, en vous l'adressant, qu'ici les paraboles P, P_1 et P_2 étant homothétiques par rapport au point O, les droites $O\gamma$, $O\gamma_1$ et $O\gamma_2$ coincident, ce qui enlève toute signification à la propriété énoncée." The Proposer has given it again, in an improved form, as a new Question.]

9795. (W. J. C. Sharp, M.A. Suggested by Quest. 5420.)—If $S_{i.j}$ denote the coefficient of t_j in $(1+t)(1+2t)\dots(1+it)$,

prove that $S_{i.j} = S_{i-1.j} + iS_{i-2.j-1} + i(i-1)S_{i-3.j-2} + \dots$

$$+ i(i-1)\dots(1-j+1)S_{i-j-1.0},$$

and that the product itself is the coefficient of x^{i+1} in the expansion of $(1-tx)^{-1/t}$ multiplied by $(i+1)!$

Solution by H. J. Woodall, A.R.C.S.

It can be easily seen that $S_{i.j} = S_{i-1.j} + iS_{i-1.j-1}$, *i.e.*, a coefficient in the given product may be thus obtained from two consecutive coefficients in the next lower product. Replace $S_{i-1.j-1}$ in a similar manner, and so on, and we get the given series. The second part follows also very easily.

12289. (Alice Gordon.)—Solve the equations

(1) $u_{x+2} + 4u_{x+1} + 4u_x = ax^2 + bx + c$; (2) $u_{x+1.y} - u_{x.y+1} + cu_{xy} = 0$

(x alone being supposed a whole number).

Solution by H. W. Curjel, B.A.; Professor Mukhopadhyay; *and others.*

Using the usual methods $(E+2)^2 u_x = ax^2 + bx + c$;

therefore $\quad u_x = (Ax+B)(-2)^x + \frac{1}{27}\{3ax^2 + (3b-4a)x + 3c - b\}$,

where A and B are arbitrary constants. Hence we have

$$\{E_x - (E_y - c)\}\, u_{xy} = 0\,; \qquad u_{xy} = (E^y - c)^x \phi(y) = (e_{d/dy} - c)_x\, \phi(y)$$

$$= \phi(x+y) - xc\phi(x+y-1) + \tfrac{1}{2}c^2x(x-1)\,\phi(x+y-2)$$
$$- \tfrac{1}{6}c^3x(x-1)(x-2)\,\phi(x+y-3) + \&c.,$$

where ϕ is an arbitrary function.

12307. (Professor Barisien.)—Etant donnée une ellipse de foyers F et F′, trouver le lieu décrit par le point P tel qu'en menant de ce point les tangentes à l'ellipse, ayant leurs points de contact en M, M′, les droites MF, M′F′ se rencontrent sur l'ellipse. Montrer que (1) ce lieu se compose des deux directrices de l'ellipse donnée et d'une ellipse, et (2)

le lieu du point de rencontre des droites M′F et MF′ est aussi une ellipse.

Solution by Professors DROZ-FARNY, SANJANA, *and others.*

Les points M et M′ décrivent sur la conique donnée deux divisions homographiques dont les points doubles sont les sommets sur l'axe focal. (CHASLES, *Sections coniques*, Art. 229.) La droite MM′ enveloppera donc une ellipse doublement tangente à la proposée aux sommets considérés. Le lieu cherché sera l'ellipse réciproque polaire de la précédente par rapport à l'ellipse donnée comme courbe directrice. (CHASLES, *loc. cit.*, Art. 473.) Elle sera donc aussi doublement tangente à la proposée aux mêmes sommets.

Comme solutions particulières on peut aussi considérer les cordes MM′ enveloppant les foyers F et F′ ce qui donne pour lieu des points P correspondants les deux directrices de l'él ipse.

Les points M et M′ décrivant deux divisions homographiques, les faisceaux F′M et FM′ sont aussi homographiques, et le lieu des points de rencontre des rayons homologues est une conique passant par les foyers F et F′.

12039. (Prof. BARISIEN.)—Un cercle (C) coupe une conique (Σ) en quatre points. Par ces quatre points on fait passer une hyperbole équilatère (H). Quel que soit le cercle (C), le rapport des distances respectives du centre de (Σ) aux centres de (C) et de (H) est constant.

Solution by W. J. GREENSTREET; Prof. KRISHNACHANDRA; *and others.*

Let $x^2/a^2+y^2/b^2-1=0$ be the conic Σ, and $x^2+y^2-2\alpha x-2\beta y+c=0$, the circle C. Then any conic through the points of intersection of C and Σ will be represented by the equation

$$x^2+y^2-2\alpha x-2\beta y+\gamma-\lambda\,(b^2x^2+a^2y^2-a^2b^2)=0.$$

This will be an equilateral hyperbola, if

$$1-\lambda b^2=\lambda a^2-1,\quad i.e.,\text{ if }\lambda=2/(a^2+b^2).$$

Then, as the centre of C is α, β, and the centre of the hyperbola is

$$\frac{\alpha}{1-\;b^2},\ \frac{\beta}{1-\lambda a^2},\quad\text{or}\quad \alpha/c^2\,(a^2+b^2),\ \ -\beta/c^2\,(a^2+b^2),$$

the ratio required is $c^2/(a^2+b^2)$, which is constant.

12266. (Professor DE WACHTER.)—Determine a point in the plane of a given ellipse such that the moment of inertia of the ellipse shall be constant for any axis passing through that point *and lying in the same plane.*

Solution by H. W. Curjel, B.A.; C. Bickerdike; *and others.*

The locus of such a point is evidently $\frac{1}{4}a^2 + x^2 = \frac{1}{4}b^2 + y^2$, where $x^2/a^2 + y^2/b^2 = 1$ is the equation to the ellipse, *i.e.*, $y^2 - x^2 = \frac{1}{4}(a^2 - b^2)$.

12300. (Professor Neuberg.)—Trouver deux nombres triangulaires, c'est-à-dire de la forme $\frac{1}{2}a(a+1)$, dont la demi-somme soit un nombre triangulaire.

Solution by Professor Droz-Farny; H. W. Curjel, B.A.; *and others.*

Il s'agit de résoudre l'équation

$$\tfrac{1}{4}x(x+1) + \tfrac{1}{4}y(y+1) = \tfrac{1}{2}z(z+1), \quad \text{ou} \quad (2x+1)^2 + (2y+1)^2 = 2(2z+1)^2.$$

La solution de cette équation indéterminée est fournie par l'identité bien connue $\quad (a^2 + 2ab - b^2)^2 + (a^2 - 2ab - b^2)^2 = 2(a^2 + b^2)^2$,
dans laquelle il suffira de prendre a et b de parité différente.

Ex. $\quad a = 4, \quad b = 1, \quad 2x+1 = 23, \quad 2y+1 = 7, \quad 2z+1 = 17,$

$$x = 11, \quad y = 3, \quad z = 8, \quad \frac{11 \cdot 12}{4} + \frac{3 \cdot 4}{4} = \frac{8 \cdot 9}{2}.$$

12338. (Professor Whitaker.)—Three lights of intensities 2, 4, 5 are placed respectively at points the coordinates of which are (0, 3), (4, 4), (9, 0); find a point in the plane of the lights equally illuminated by all of them.

Solution by H. W. Curjel, B.A.; M. Brierley; *and others.*

The required point is evidently given by

$$\frac{2}{x^2 + (y-3)^2} = \frac{4}{(x-4)^2 + (y-5)^2} = \frac{5}{(x-9)^2 + y^2},$$

and is therefore $(2, -1)$ or $(-6, -5)$.

12274. (Professor Niewenglowski.)—Trouver la limite de l'expression $L \equiv \frac{1}{2}\tan\frac{1}{3}a + \frac{1}{3}\cot\frac{1}{3}a$, pour $a = 3\pi$.

Solution by H. W. Curjel, B.A.; Professor Beyens; *and others.*

$$\text{Limit of } L = \text{limit of } \tfrac{5}{36}\,\frac{2\cos\frac{1}{3}a\cos\frac{1}{2}a - 3\sin\frac{1}{3}a\sin\frac{1}{2}a}{-2\cos\frac{1}{3}a\sin\frac{1}{2}a - \frac{13}{6}\sin\frac{1}{3}a\cos\frac{1}{2}a}$$

$$= \frac{5}{36} \times \frac{0}{-2} = 0.$$

3117. (Rev. T. R. Terry, M.A.)—A rod is placed with one extremity at the middle point of the line joining two centres of force, which attract inversely as the square of the distance, the rod being at right angles to this line: find the velocity with which the centre of the rod will cross this line. If the rod were placed at right angles to the line joining the centres, and very near the position of equilibrium, determine the time of its oscillation about that position.

Solution by H. W. Curjel, M.A.; Prof. Sanjána; *and others.*

Let the distance between the centres of force $= 2a$. Attraction on a particle m at distance x from the line joining the centres of force and equidistant from them $= -\dfrac{2\mu xm}{(x^2+a^2)^{\frac{3}{2}}}$. Therefore potential energy of the particle

$$= 2\mu m\left\{\frac{1}{a}-\frac{1}{(x^2+a^2)^{\frac{1}{2}}}\right\};$$

hence the potential energy of the rod of length b and mass M in its initial position

$$= \frac{2\mu M}{b}\left\{\frac{b}{a}-\log\frac{b+(b^2+a^2)^{\frac{1}{2}}}{a}\right\};$$

hence, if v = velocity required,

$$v^2 = \frac{4\mu}{b}\left\{\frac{b}{a}-\log\frac{b+(b^2+a^2)^{\frac{1}{2}}}{a}\right\}.$$

12556. (H. J. Woodall, A.R.C.S.)—Solve the following equations:—

(1) $x^6-6x^5+25x^4-58x^3+102x^2-96x+72=0$;

(2) $x^6-10x^5+64x^4-231x^3+616x^2-1000x+1120=0$;

(3) $x^8-10x^7+57x^6-208x^5+552x^4-1048x^3+1472x^2-1344x+768=0$.

Solution by Rev. T. P. Kirkman, F.R.S.; Professor Aiyar; *and others.*

The quadratic factors of (1) and (2) are

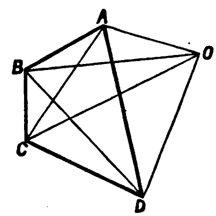
$$(x^2-x+2)(x^2-2x+6)(x^2-3x+6)=0,$$

$$(x^2-x+7)(x^2-4x+8)(x^2-5x+20)=0.$$

The answerer has found it more amusing to continue his search for a method of solution briefer and less tentative than the one he yet knows than to complete the tables now necessary for the conquest of the two equations of the 10th and 12th degrees; and he believes that he can find it. At present he will be thankful to see proposed a few more equations U_{2m} of degrees below the ninth, all having for second coefficient $A \ngtr 40$, and all having in the last coefficient L any number $\ngtr 10$ of prime factors: $(x-1)\,U_{2m-1} = U_{2m}$.

10192. (Professor Décamps.)—Si l'on joint les sommets d'un quadrilatère convexe à un point extérieur, pris à volonté dans son plan, le produit des aires des triangles qui ont pour bases les diagonales est égal à la somme des produits des aires des triangles ayant pour bases les côtés opposés. Qu'arrive-t-il si le point est pris à l'intérieur du quadrilatère?

Solution by H. J. Woodall, A.R.C.S.; Prof. Sanjana; *and others.*



Let ABCD be the quadrilateral, O the exterior point. Let AOB = α, BOC = β, COD = γ, DOA = δ.

Then, the difference between the product of the triangles on the diagonals and the sum of the products of the triangles on opposite sides (all the vertices being at O) is

$$\text{OAC} \times \text{OBD} - \text{OAB} \times \text{OCD} - \text{OBC} \times \text{OAD}$$
$$= \tfrac{1}{4}\text{OA} \,.\, \text{OB} \,.\, \text{OC} \,.\, \text{OD} \left\{\sin(\alpha+\beta)\sin(\beta+\alpha) - \sin\alpha\sin\gamma - \sin\beta\sin\delta\right\}.$$

In the figure DOA = δ is measured in the negative direction; it is equal to $\alpha+\beta+\gamma$; substitute in the above formula, and it vanishes.

The different positions of O affect the result in such way that, when O crosses one of the six lines AB, BC, CD, DA, AC, BD, the corresponding triangle has its sign changed. Thus we are able to account for all variations in the formula.

3365. (S. Watson.)—From any point P in the circumference of a given circle a chord PQ is drawn at random; and upon it two points R, S are taken at random. What is the chance that the circle upon RS as diameter will be wholly upon the given circle?

Solution by D. Biddle.

Let θ = angle formed by chord with diameter at P, x = distance of R from mid-point of chord, u, v = respective distances of the two positions of S (from mid-point of chord) when the circles touch. Then $\dfrac{u-v}{2\cos\theta}$ = the chance for any position of R on any given chord. The limits of the centre of the smaller circle, defined by $\frac{1}{2}(u-v)$, can at any time be found by describing an ellipse having R and O (the centre of the given circle) as its foci, and its major axis equal to the radius.

Now $\qquad u = \dfrac{x+\cos^2\theta}{1+x}, \quad$ and $\quad v = \dfrac{x-\cos^2\theta}{1-x},$

whence $\qquad \dfrac{u-v}{2\cos\theta} = \dfrac{\cos^2\theta - x^2}{\cos\theta\,(1-x^2)}.$

Consequently $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}\int_0^{\cos\theta} \dfrac{\cos^2\theta - x^2}{\cos^2\theta\,(1-x^2)}\, d\theta \,.\, dx$ = probability required.

This becomes $\dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}\left\{\left(1-\frac{1}{\cos^2\theta}\right)\tfrac{1}{2}\log\frac{1+\cos\theta}{1-\cos\theta}+\cos\theta\right\}d\theta,$

which does not readily yield to further integration. We can, however, approximate to the value by observing that

$$\tfrac{1}{2}\log\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\dots.$$

Thus $\left(1-\dfrac{1}{\cos^2\theta}\right)\tfrac{1}{2}\log\dfrac{1+\cos\theta}{1-\cos\theta}+\dfrac{1}{\cos\theta}=2\left(\dfrac{\cos\theta}{1.3}+\dfrac{\cos^3\theta}{3.5}+\dfrac{\cos^5\theta}{5.7}+\dots\right),$

and we have

$$\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{\cos\theta}{1.3}+\frac{\cos^3\theta}{3.5}+\dots\right)d\theta=\frac{4}{\pi}\left(\frac{1}{1.3}\cdot\frac{1}{1}+\frac{1}{3.5}\cdot\frac{2}{3}+\frac{1}{5.7}\cdot\frac{8}{15}+\dots\right).$$

The second factor in the several terms represents the result of integrating the particular odd power of $\cos\theta$, and is in reality compounded of increasing numbers of simple factors

$$1,\quad \tfrac{2}{3},\quad \tfrac{2}{3}\cdot\tfrac{4}{5},\quad \tfrac{2}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{7},\quad \tfrac{2}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{7}\cdot\tfrac{8}{9},\quad \&c.$$

The several whole terms to be summed are each of them products of n terms of the following series:—

$$\underset{(1)}{\frac{1}{3}},\ \underset{(2)}{\frac{1.2}{3.5}},\ \underset{(3)}{\frac{3.4}{5.7}},\ \underset{(4)}{\frac{5.6}{7.9}},\ \underset{(5)}{\frac{7.8}{9.11}},\ \underset{(6)}{\frac{9.10}{11.13}},\ \underset{(7)}{\frac{11.12}{13.15}},\ \underset{(8)}{\frac{13.14}{15.17}},\ \underset{(9)}{\frac{15.16}{17.19}},$$

$$\underset{(10)}{\frac{17.18}{19.21}},\ \underset{(11)}{\frac{19.20}{21.23}},\ \underset{(12)}{\frac{21.22}{23.25}},\ \&c.$$

Consequently we are able to render the required probability as follows:—

$$\frac{4}{\pi}\left\{\tfrac{1}{3}+\Sigma_1^{\infty}\frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)^2\,(2n+3)}\right\};$$

or, if the notation be allowed,

$$\frac{4}{\pi}\left\{\tfrac{1}{3}+\Sigma_1\frac{n!}{(n+\frac{1}{2})!\,\{(2n+2)^2-1\}}\right\},$$

in which $\dfrac{n!}{(n+\frac{1}{2})!}=\dfrac{\text{continuous product of even numbers to } 2n}{\text{continuous product of odd numbers to } (2n+1)}.$

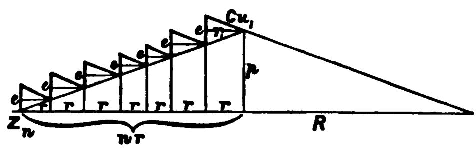
Summing 5, 10, 15, 20 terms of the original series, we get ·514867, ·524058, ·526528, ·527590.

Roughly, the probability is ·53, or more than $\frac{1}{2}$.

12524. (J. O'Byrne Croke, M.A.)—If we arrange the cells of a battery in series, the area included between the curve of potential and a base line through the position of the last zinc is equal to $\frac{1}{2}ne(\mathrm{R}+r)$; r being the resistance of a single cell and e its electromotive force, n the whole number of cells, and R the whole electrical resistance.

Solution by the PROPOSER; Professor CHAKRIVARTI; *and others.*

The area of the curve of potential is equal to the sum of the areas of the n triangles of which the perpendicular of each is r and the base e, plus



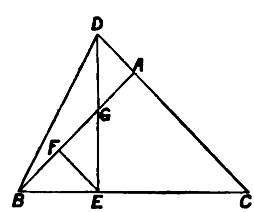
the area of the triangle of which the perpendicular is p, and the base $R + nr$. That is, area $= \frac{1}{2}p(R + nr) + \frac{1}{2}ner$.

But $\qquad R : p :: R + nr : ne; \qquad \therefore\ p = \dfrac{neR}{R + nr};$

and $\qquad \therefore\ \text{area} = \frac{1}{2}(R + nr)\dfrac{neR}{R + nr} + \frac{1}{2}ner = \frac{1}{2}ne(R + r).$

12513. (Professor LACORD.)—Dans un triangle rectangle isoscèle, démontrer que le côté est inférieur à trois fois l'excès de l'hypoténuse sur ce côté.

Solution by W. J. DOBBS, M.A.; T. SAVAGE; *and others.*



Let ABC be a right-angled isosceles triangle, A being the right angle. Produce CA to D, making AD $= \frac{1}{2}$AC, and draw DE perpendicular to BC, meeting AB in G, and let F be the mid-point of BG. Then DGA and BGE are right-angled isosceles triangles. Hence FE is perpendicular to BG; therefore

$$GE > FG > GA;$$

and therefore $\quad BG > GD.$

Therefore $\angle BDG > GBD$; therefore compt. DBE < compt. BDA; therefore DC < BC; therefore DA < BC − AC; *i.e.*, a third of a side is less than the excess of the hypotenuse over a side.

[Mr. BIDDLE puts the solution thus:—The side being unity, the excess of the hypotenuse over it $= \sqrt{2} - 1$. Deducting this from the side, we have $2 - \sqrt{2}$. But $2 - \sqrt{2} < 2(\sqrt{2} - 1)$, because $4 < 3\sqrt{2}$, that is, $\sqrt{(16)} < \sqrt{(18)}$. Hence the theorem follows.]

12559. (Rev. T. ROACH, M.A.)—Prove that

$$1-x^6/6!+x^{12}/12!-\ldots = \tfrac13\left\{\cos x+2\cosh\tfrac12\sqrt{3}x\cos\tfrac12 x\right\};$$
$$1-x^3/3!+x^6/6!-\ldots = \tfrac13\left(e^{-x}+2e^{\frac12 x}\cos\tfrac12\sqrt{3}x\right);$$
$$x-x^4/4!+x^7/7!-\ldots = \tfrac13\left\{2e^{\frac12 x}\sin\left(\tfrac16\pi+\tfrac12\sqrt{3}x\right)-e^{-x}\right\}.$$

Solution by H. W. CURJEL, M.A., *and* Rev. D. T. GRIFFITHS, M.A.

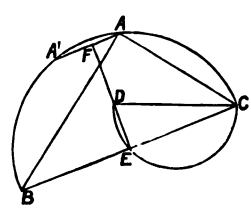
$$\begin{aligned}
\text{The first series} &= \tfrac13(\cos x+\cos\omega x+\cos\omega^2 x), \text{ where } \omega^2+\omega+1=0,\\
&= \tfrac13\left\{\cos x+2\cos\tfrac12(\omega+\omega^2)x\cos\tfrac12(\omega^2-\omega)x\right\}\\
&= \tfrac13\left\{\cos x+2\cos\tfrac12 x\,.\,\tfrac12\left(e^{\frac12\sqrt3 x}+e^{-\frac12\sqrt3 x}\right)\right\}\\
&= \tfrac13\left(\cos x+2\cos\tfrac12 x\cosh\tfrac12\sqrt3\,x\right).
\end{aligned}$$

$$\begin{aligned}
\text{Second series} &= \tfrac13\left(e^{-x}+e^{-\omega x}+e^{-\omega^2 x}\right)\\
&= \tfrac13\left\{e^{-x}+e^{\frac12 x}\left(e^{\frac12(i\sqrt3 x)}+e^{\frac12(-i\sqrt3 x)}\right)\right\}\\
&= \tfrac13\left\{e^{-x}+2e^{\frac12 x}\cos\tfrac12\sqrt3\,x\right\}.
\end{aligned}$$

$$\begin{aligned}
\text{Third series} &= \tfrac13\left\{-e^{-x}-\omega^2 e^{-\omega x}-\omega e^{-\omega^2 x}\right\}\\
&= \tfrac13\left\{-e^{-x}+e^{-\frac13 i\pi}e^{\frac12 x+i\frac12\sqrt3 x}+e^{\frac13 i\pi}e^{\frac12(x-i\sqrt3 x)}\right\}\\
&= \tfrac13\left\{-e^{-x}+2e^{\frac12 x}\cos\left(-\tfrac13\pi+\tfrac12\sqrt3\,x\right)\right\}\\
&= \tfrac13\left\{2e^{\frac12 x}\sin\left(\tfrac16\pi+\tfrac12\sqrt3\,x\right)-e^{-x}\right\}.
\end{aligned}$$

12564. (I. ARNOLD.)—Given the difference of the legs, and also the difference between the hypotenuse and the perpendicular upon it from the right angle, to construct the triangle.

Solution by D. BIDDLE; C. BICKERDIKE; *and others.*



On CD, the given difference between the hypotenuse and the perpendicular, describe the semicircle CED, and make DE = the given difference between the legs. Join CE and produce, making EB = CD.

On BC describe the semicircle BAC. Produce ED, and make EF = EC. Finally, draw FA parallel to BC, to meet the semicircle in A, and join AB, AC. ABC is the triangle required. For, by analysis,

$$BC = EF + CD, \quad\text{and}\quad AB = AC + DE.$$

Moreover, $\quad EF\,.\,BC = AB\,.\,AC, \quad$ and $\quad AB^2+AC^2 = BC^2,$

whence we get $\quad EF = (CD^2-DE^2)^{\frac12}.$

But, by construction, $BE = CD$, and $EF = CE = (CD^2-DE^2)^{\frac12}$.

Hence, &c.

12507. (Professor BEVERAGE.) — Of the quadratic $x^2 - ax + b = 0$, find the mean values of the roots, which are known to be real, b being unknown and positive.

Solution by D. BIDDLE; H. J. CURJEL, M.A.; *and others.*

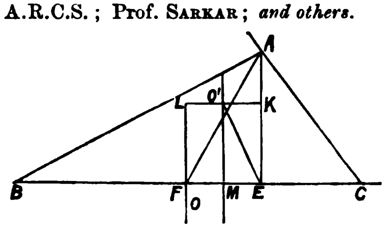
$x = \frac{1}{2}\left\{a \pm (a^2 - 4b)^{\frac{1}{2}}\right\}$, and, since the roots are real, b may range from 0 to $\frac{1}{4}a^2$. Consequently the mean values are represented by

$$\tfrac{1}{2}\left\{a \pm \frac{4}{a^2}\int_0^{\frac{1}{4}a^2}(a^2-4b)^{\frac{1}{2}}\,db\right\} = \tfrac{1}{2}\left\{a \pm \frac{4}{a^2}\cdot\frac{a^3}{6}\right\} = \tfrac{1}{2}\left(1 \pm \tfrac{2}{3}\right)a.$$

The mean values therefore are $\frac{5}{6}a$ and $\frac{1}{6}a$.

10228. (Professor MASCART.)—Etant donnés, dans un plan, deux droites AE, AF et un point O′, construire un triangle ABC, sachant que la bissectrice et la hauteur issues de A sont dirigées suivant AE, AF, et que O′ est le centre du cercle des neuf points.

Solution by H. J. WOODALL, A.R.C.S.; Prof. SARKAR; *and others.*



Draw O′K perpendicular to AE (the perpendicular), to meet AE in K; produce KO′ and make O′L = KO′.

Draw LF parallel to AE to meet AF in F; draw FE perpendicular to AE and produce FE both ways. Join O′E, O′F (the radius of the N.P. circle); twice O′E is the radius of the circumcircle which passes through A, and has its centre in LF; thence draw this circle cutting EF produced in B, C.

Join AB, AC. ABC is the required triangle.

12520. (EDITOR.)—Find three square numbers in arithmetical progression, such that the square root of each increased by unity shall be a square.

Solution by Rev. D. T. GRIFFITHS; M. BRIERLEY; *and others.*

Let $(x^2-1)^2$, $(y^2-1)^2$, $(z^2-1)^2$ be the numbers; then we have

$$(x^2-1)^2 \sim (y^2-1)^2 = (y^2-1)^2 \sim (z^2-1)^2;$$

that is, $$(x^2+y^2-2)(x^2-y^2) = (y^2+z^2-2)(y^2-z^2).$$

This is equivalent to $x^2+y^2-2 = a(y^2+z^2-2)$, $a(x^2-y^2) = y^2-z^2 \ldots (1)$.

General expressions for square numbers in A.P.

$$(2rs-r^2+s^2)^2, \quad (r^2+s^2)^2, \quad (2rs+r^2-s^2)^2;$$

in the particular case $r = 2$, $s = 1$, we have the numbers 1, 25, 49;

$$\therefore \quad x^2-1 = 1, \quad y^2-1 = 5, \quad z^2-1 = 7.$$

Using these particular values in equations (1), the constant a is known, *i.e.*, $a = \frac{1}{2}$.

Eliminating z^2 between equations (1), we have $5x^2-y^2 = 4$.

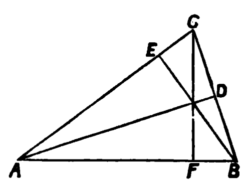
Solution: $\quad x = 5, \quad y = 11; \quad \therefore z = 13;$

hence required numbers are 576, 14,400, 28,224.

[Mr. Davis sends the following solution:—If three numbers are such that their squares are in A.P., they must be of the form $x^2-2px-p^2$, x^2+p^2, $x^2+2px-p^2$. The simplest particular case is 1, 5, 7; and one is naturally in the first place led to inquire whether it is possible for $m+1$, $5m+1$, $7m+1$ to be each of them a square. By inspection, or otherwise, $m = 24$ satisfies these conditions. Thus the numbers required are the squares of 24, 120, 168; *i.e.*, 576, 14400, 28224.]

12582. (Professor Droz-Farny.)—Dans un triangle isoscèle un des côtés égaux AB est fixe: les deux autres côtés tournent autour de leurs sommets respectifs. Chercher les enveloppes des côtés du triangle orthique. [Le triangle orthique a pour sommets les pieds des hauteurs.]

Solution by H. W. Curjel, M.A.; Professor Chakrivarti; *and others.*



Let DEF be the pedal triangle. Take A as origin, and AB as axis of x. Let

$$AB = AC = b \quad \text{and} \quad \angle DAB = \theta.$$

Then the coordinates of D, E, F are $b\cos^2\theta$, $b\cos\theta\sin\theta$; $b\cos^2 2\theta$, $b\cos 2\theta\sin 2\theta$; $b\cos 2\theta$, 0. Equation to EF is

$$yt^3+t^2(x+b)+yt+(x-b) = 0,$$

where $t = \tan\theta$, and the equation to DF is the same with the sign of t changed.

Hence the envelope of EF and DF is

$$y^2(4x-5b)^2 = (3x^2-y^2-3b^2)\left\{3y^2-(x+b)^2\right\},$$

i.e., $$(x^2+y^2)^2-14xy^2+2x^3b+11y^2b^2-2b^3x-b^4 = 0,$$

a bicircular quartic.

Equation to DE is

$$x\cos 3\theta+y\sin 3\theta = b\cos 2\theta\cos\theta, \quad \text{or} \quad t^3y+(3x-b)t^2-3yt+b-x = 0;$$

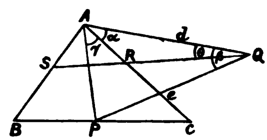
therefore envelope of DE is

$$3b^2y^2 = \left\{3y^2-(3x-b)(x-b)\right\}\left\{9y^2-(3x-b)^2\right\},$$

i.e., $$27(x^2-y^2)^2-54bx(x^2-y^2)+36x^2b^2-15y^2b^2-10xb^3+b^4 = 0.$$

1999. (R. Tucker, M.A.)—(1) P is a given point on the side of a triangle, and Q another given point in the same plane: it is required to inscribe in the triangle a maximum triangle having P for a vertex and its base passing through Q. Again, (2) P is a given point on a circle, and Q a point in the same plane with it: it is required to inscribe in the circle a maximum triangle having P for its vertex and its base passing through Q. Discuss the several cases fully.

Solution by H. J. Woodall, A.R.C.S.



1. Let ABC be the triangle. Let $AQ = d$, $QP = e$, angles $CAQ = \alpha$, $AQP = \beta$, $PAC = \gamma$, $AQR = \theta$, where RS is the required base (through Q),

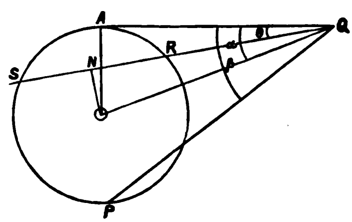
$$RS = QS - QR = d\sin(\alpha + A)/\sin(\alpha + A + \theta) - d\sin\alpha/\sin(\alpha+\theta)$$
$$= d\sin A\sin\theta/\sin(\alpha+\theta)\sin(\alpha+A+\theta).$$

Perpendicular from P on $RS = e\sin(\beta-\theta)$. Therefore

$$u = \text{area of triangle} = \tfrac{1}{2}e\sin(\beta-\theta)\,d\sin A\sin\theta/\sin(\alpha+\theta)\sin(\alpha+A+\theta).$$

The maximum and minimum values of this expression can be obtained from the equation

$$\cos A\sin(\beta-2\theta) + \cos\beta\sin(A+2\alpha+2\theta) - \sin(A+2\alpha+\beta) = 0.$$



2. Let $AO = a$, $AQ = d$, $QP = e$, angles $AQR = \theta$, $AQO = \alpha$, $AQP = \beta$.

Then we have

$$QO = d\sec\alpha,$$
$$QN = d\sec\alpha\cos(\alpha-\theta).$$
$$d^2 = QR\,.\,QS = QN^2 - NR^2;$$
$$\therefore\quad NR = \{QN^2 - d^2\}^{\frac{1}{2}}$$
$$= d\sec\alpha\left\{\cos^2(\alpha-\theta) - \cos^2\alpha\right\}^{\frac{1}{2}}.$$

Perpendicular from P on $RS = e\sin(\beta-\theta)$; therefore

$$u = \text{area of triangle} = e\sin(\beta-\theta)\,d\sec\alpha\left\{\cos^2(\alpha-\theta) - \cos^2\alpha\right\}^{\frac{1}{2}}.$$

The maximum and minimum values of this expression can be obtained from the equation

$$\cos(\beta-\theta)\cos^2\alpha - \cos(\alpha-\theta)\cos(\alpha+\beta-2\theta) = 0.$$

12531. (S. Tebay, B.A.)—Give a simple method of finding *ad libitum* n square integers whose sum is a square.

Solution by A. Martin, LL.D.; Rev. D. T. Griffiths, B.A.; *and others.*

Take the identity $(w+z)^2 = w^2+2wz+z^2 = (w-z)^2+4wz$ .... (1).

Now, if we can transform $4wz$ into a square, we shall have *two* square numbers whose sum is a square. This will be effected by taking $w = p^2$, $z = q^2$, for then $4wz = 4p^2q^2 = (2pq)^2$, and we have

$$(p^2+q^2)^2 = (p^2-q^2)^2+(2pq)^2 \quad \text{.....................} \; (2).$$

In (1), put $x+y$ for w, and we have

$$\begin{aligned}(x+y+z)^2 &= (x+y-z)^2+4(x+y)z = (x+y-z)^2+4xz+4yz \\ &= (x+z-y)^2+4xy+4yz = (y+z-x)^2+4xy+4xz.\end{aligned}$$

Assuming $x = p^2$, $y = q^2$, $z = r^2$, we have

$$\begin{aligned}(p^2+q^2+r^2)^2 &= (p^2+q^2-r^2)^2+(2pr)^2+(2qr)^2 \\ &= (p^2+r^2-q^2)^2+(2pq)^2+(2qr)^2 \\ &= (q^2+r^2-p^2)^2+(2pr)^2+(2pq)^2 \quad \text{............} \; (3).\end{aligned}$$

These are three sets of *three* squares, the sum of each of which sets $= (p^2+q^2+r^2)^2$, where p, q, r may have any integer values.

We can in the same way obtain from (1) any number n of squares whose sum is a square by simply substituting for w the sum of two, of three, of four, ... $n-1$ other quantities, thus:—

Putting $a_1+a_2+a_3+a_4+\ldots+a_{n-1} = w$, $a_n = z$, we have

$$\begin{aligned}(a_1+a_2+a_3+\ldots+a_n)^2 = {} & (a_1+a_2+a_3+\ldots+a_{n-1}-a_n)^2 \\ & + 4(a_1+a_2+a_3+\ldots+a_{n-1})a_n.\end{aligned}$$

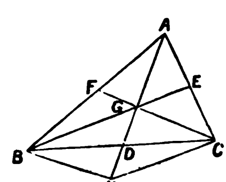
Now, taking $a_1 = b_1^2$, $a_2 = b_2^2$, $a_3 = b_3^2$, ... $a_n = b_n^2$, we have

$$\begin{aligned}(b_1^2+b_2^2+b_3^2+\ldots+b_n^2)^2 = {} & (b_1^2+b_2^2+b_3^2+\ldots+b_{n-1}^2-b_n^2)^2+(2b_1b_n)^2 \\ & +(2b_2b_n)^2+(2b_3b_n)+\ldots+(2b_{n-1}b_n)^2 \ldots \; (4),\end{aligned}$$

one set of n square numbers whose sum is a square, and we can obtain by cyclic permutation $n-1$ other sets, the sum of each of which is equal to $(b_1^2+b_2^2+b_3^2+\ldots+b_n^2)^2$, where b_1, b_2, b_3, b_4, ... b_n may have any integer values chosen at pleasure.

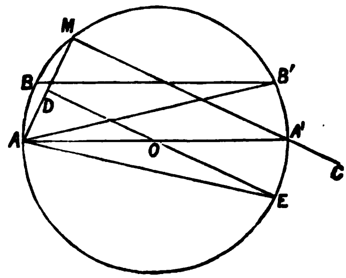
12550. (Professor A. E. A. Williams.)—Given the bisectors, form the triangle.

Solution by C. W. Bourne, M.A.; Professor Sanjána; *and others.*



Let l, m, n be the lengths of the bisectors. Construct a triangle GBH whose sides are $\frac{2}{3}l$, $\frac{2}{3}m$, $\frac{2}{3}n$ in length; and complete the parallelogram GBHC; produce HG to A, making AG = GH. Then ABC is easily seen to be the required triangle.

12475. (Professor Morel.) — Etant donnée une circonférence de diamètre AA′, on mène une corde quelconque BB′ parallèle à ce diamètre; on prend la corde AM double de la distance des deux parallèles AA′, BB′; démontrer que l'angle AA′C est double de l'angle AA′B′, C étant un point quelconque situé sur le prolongement de MA′.

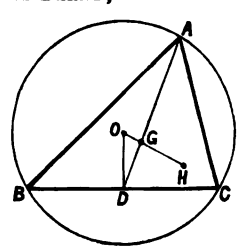


Solution by T. Savage; H. W. Curjel, M.A.; *and others.*

Joining the middle point D of AM to the centre O of the circle, and producing DO to meet the circle in E, we see that the triangles ADE, A′B′A are equal in all respects; therefore ∠ MA′A is double ∠ B′AA′; therefore ∠ AA′C is double ∠ B′A′A.

12549. (Professor Neuberg.)—On donne l'orthocentre H d'un triangle ABC et le centre O du cercle circonscrit. Le sommet A décrit une droite ou une circonférence; démontrer que le côté BC enveloppe une conique ayant O pour foyer.

Solution by W. J. Dobbs, M.A.; Professor Droz-Farny; *and others.*



Let the median AD meet OH in G.

Then G is a fixed point (for it divides OH in the ratio 1 : 2), and

$$DG : GA = 1 : 2;$$

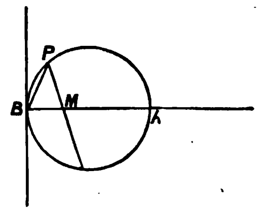
therefore locus of D is similar to locus of A.

Thus, the pedal of the envelope of BC with respect to the fixed point O is either (i.) a straight line, or (ii.) a circle.

Therefore the envelope of BC is either (i.) a parabola with focus O, or (ii.) a conic with focus O.

10189. (Professor De Longchamps.)—Soit une circonférence Δ; on prend, dans ce cercle, un diamètre fixe AB. Par un point M, mobile sur Δ, on trace une droite rencontrant AB en P et telle que PMB soit un triangle isoscèle :—(1) Démontrer que le lieu des centres des cercles circonscrits à PMB est une strophoïde; (2) Trouver le lieu décrit par le pôle de MP, et construire ce lieu, qui est une cubique unicursale; (3) Trouver l'enveloppe de la droite PM.

Solution by **H. J. Woodall, A.R.C.S.**



1. The equation to the circle is

$$x^2 - 2ax + y^2 = 0.$$

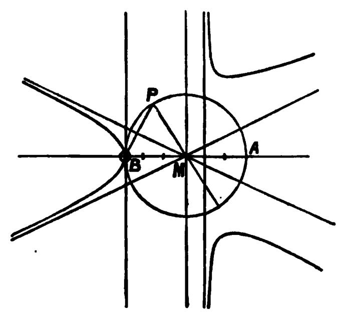
Let P be $(x_1 y_1)$; M is $(2x_1, 0)$. The circle round PMB is

$$x^2 - 2xh + y^2 - 2yk = 0,$$

where $h = x_1,$

$$k = (a - x_1)\, x_1 (2ax_1 - x_1^2)^{-\frac{1}{2}},$$

whence, putting current coordinates for (h, k), we get $y^2 (2a - x) = x (a - x)^2$.



2. PM is $y_1 (x - 2x_1) + x_1 y = 0$, whence we find the locus of its pole to be

$$y^2 (3x - 4a) = x (x - a)^2.$$

This locus has a conjugate point at $(a, 0)$; its asymptotes are

$$3x - 4a = 0,$$

and $y\sqrt{3} = \pm (x - a)$;

the minimum value of y is where

$$x = a (1 + \tfrac{1}{3}\sqrt{3}),$$

whence $y = a (3 + 2\sqrt{3})^{\frac{1}{2}}$

$= 2{\cdot}6a$ about.

The form of the curve at $(0, 0)$ is parabolic.

3. PM is $y_1 (x - 2x_1) + x_1 y = 0$. Since $(x_1 y_1)$ is on $x^2 - 2ax + y^2 = 0$, we must substitute $y_1^2 = 2ax_1 - x_1^2$, whence we get

$$4x_1^3 - 4x_1^2 (2a + x) + x_1 (x^2 + 8ax + y^2) - 2ax^2 = 0.$$

The envelope of this is found to be

$$\begin{vmatrix} y^2 + 4ax - 4a^2, & 3ax^2, & 2ax^2 (2a + x) \\ 4 (2a + x), & 2 (x^2 + 8ax + y^2), & 24ax^2 \\ 3, & 2 (2ax + x), & x^2 + 8ax + y^2 \end{vmatrix} = 0.$$

3029. (S. Watson.)—Find the locus of the intersection of normals at the extremities of focal chords in an ellipse.

Solution by Professor **Lampe, LL.D.**

Taking the polar equation of *any* conic $r = p/(1 - \epsilon \cos\phi)$ or $y^2 = x^2 (\epsilon^2 - 1) + 2p\epsilon x + p^2$, we find

$$dy/dx = \{p\epsilon + x (\epsilon^2 - 1)\}/y = (\epsilon - \cos\phi)/\sin\phi\,;$$

hence the equation of the normal in a point P of the conic will be

$$y (\cos\phi - \epsilon) - x \sin\phi = -p\epsilon \sin\phi / (1 - \epsilon \cos\phi) \;\ldots\ldots\ldots\ldots\; (1).$$

Changing ϕ into $\phi+\pi$, we get the equation of the normal at the other extremity of the focal chord,

$$y(\cos\phi+\epsilon)-x\sin\phi = -p\epsilon\sin\phi/(1+\epsilon\cos\phi) \text{ .... } (2).$$

The equation of the locus required is to be obtained by eliminating ϕ between (1) and (2). Combining (2)+(1) and (2)−(1), we have

$$y\cos\phi-x\sin\phi = -p\epsilon\sin\phi/(1-\epsilon^2\cos^2\phi) \text{ .... } (3),$$

$$y = p\epsilon\sin\phi\cos\phi/(1-\epsilon^2\cos^2\phi) \text{ .... } (4).$$

Substituting (4) in (3), we find

$$\cos^2\phi = (x-p\epsilon)/(p\epsilon+x\epsilon^2),\quad \sin^2\phi = \{2p\epsilon+x(\epsilon^2-1)\}/(p\epsilon+x\epsilon^2) \text{...... } (5).$$

(3) gives $\quad y^2\cos^2\phi = x^2\sin^2\phi\,\{x-p\epsilon/(1-\epsilon^2\cos^2\phi)\}^2$

$$= x^2\sin^2\phi\,\{(x-p\epsilon)/(1+\epsilon^2)\}^2,$$

or $\quad y^2(1+\epsilon^2)^2 = \{2p\epsilon+x(\epsilon^2-1)\}\{x-p\epsilon\},$

or $\quad y^2(1+\epsilon^2)^2+x^2(1-\epsilon^2)-xp(3\epsilon-\epsilon^3)+2p^2\epsilon^2 = 0.$

This is the equation of the locus, ellipse, parabola, hyperbola, if the conic given be of the same kind.

Putting $x_0 = p(3\epsilon-\epsilon^3)/2(1-\epsilon^2)$, and $x = x_0+x'$, (6) becomes

$$\frac{4y^2(1-\epsilon^2)}{p^2\epsilon^2}+\frac{4x'^2(1-\epsilon^2)^2}{p^2\epsilon^2(1+\epsilon^2)^2} = 1 \text{ .... } (7).$$

$\epsilon>1$ (hyperbola); semiaxes: $a_1 = \frac{1}{2}\frac{p\epsilon(\epsilon^2+1)}{\epsilon^2-1},\quad b_1 = \frac{1}{2}\frac{p\epsilon}{\epsilon^2-1} = \frac{1}{2}c.$

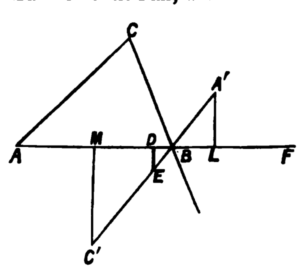
$\epsilon<1$ (ellipse); semiaxes: $a_2 = \frac{1}{2}\frac{p\epsilon(\epsilon^2+1)}{1-\epsilon^2},\quad b_2 = \frac{1}{2}\frac{p\epsilon}{1-\epsilon^2} = \frac{1}{2}c.$

The axis of the locus perpendicular to the focal axis is equal to half the distance of the two foci of the conic section given.

[For another Solution, see Vol. LXII., p. 40.]

12436. (R. Knowles, B.A.)—On AB, a side of a triangle ABC, AD is taken $= \frac{1}{2}(AB+BC)$; prove that the perpendicular from D on AB bisects the line joining the centres of the escribed circles touching AB and BC.

Solution by Professors Sanjána, M.A., Krishnachandra, *and others.*



Let C′, A′ be the ex-centres, and C′M, A′L perpendiculars to AB. Now the difference between C′M and A′L $= r_3-r_1$

$$= \frac{s}{s-c}-\frac{s}{s-a} = \frac{(c-a)s}{(s-a)(s-c)}$$

$$= (c-a)\cot\tfrac{1}{2}B,$$

and $\quad DE = DB\cot\frac{1}{2}B$

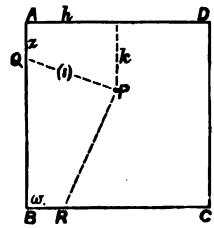
$$= \tfrac{1}{2}(c-a)\cot\tfrac{1}{2}B;$$

therefore $DE = \frac{1}{2}(c'M-A'a)$;

therefore DE bisects Ac'.

9328. (W. J. Greenstreet, B.A.)—ABCD is a square field. A path runs round AB, BC; find where, on starting from A, I must leave the path AB to get to P in the shortest time, P being any point on the field, and my rates of walking x miles per hour and y miles per hour on path and field respectively. Where must P be to be reached most quickly (1) by a straight course; (2) keeping on AB and leaving at a point on BC.

Solution by H. J. Woodall, A.R.C.S.



1. Take the coordinates of P as in figure; then, leaving AB at a point Q, where AQ = z,

$$u = z/x + \{h^2 + (k-z)^2\}^{\frac{1}{2}}/y,$$

$$du/dz = 1/x + (z-k)/y\,\{h^2 + (z-k)^2\}^{\frac{1}{2}} = 0,$$

if $$k - z = hy/(x^2-y^2)^{\frac{1}{2}};$$

$$z = 0 \quad \text{if} \quad k = hy/(x^2-y^2)^{\frac{1}{2}}$$

connects x, y, h, k for a straight course; if $k < hy/(x^2-y^2)^{\frac{1}{2}}$, the course would be straight; the interpretation would require walking backward.

2. Leaving BC at a point R where BR = w; therefore

$$u = w/x + \{(a-k)^2 + (w-h)^2\}^{\frac{1}{2}}/y,$$

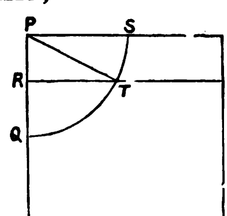
$$du/dw = 1/x + (w-h)/y\,\{(a-k)^2 + (w-h)^2\}^{\frac{1}{2}} = 0,$$

if $$h - w = y\,(a-k)/(x^2-y^2)^{\frac{1}{2}};$$

therefore $w = 0$, if $\quad h \not> y\,(a-k)/(x^2-y^2)^{\frac{1}{2}}$.

12476. (Professor Shields.)—A square field M has a narrow path R running across it parallel with one side, cutting off one-quarter of its area. A horse is tied *inside* to the corner post P of the field with a rope equal in length to one-half of the side of the field, and can graze over just one acre on the opposite side of the path from him. Find the area of the field.

Solution by T. Savage; Morgan Brierley; *and others.*



Denoting side of square by s,

$$\text{sector PQT} = \tfrac{1}{6} \cdot s^2 \cdot \tfrac{1}{4}\pi,$$

$$\Delta\text{PRT} = \tfrac{1}{32} \cdot s^2 \cdot \sqrt{3};$$

therefore $\quad s^2\,(\tfrac{1}{6}\pi - \tfrac{1}{8}\sqrt{3}) = 1$ acre.

Hence the area of M is

$$\frac{96}{4\pi - 3\sqrt{3}} \text{ acres.}$$

12305. (Professor RAMACHANDRA ROW.) — Let $p_1, p_2, p_3, \ldots$ be any numbers whose product is π. Choose P_1 a multiple of π/p_1, so that $P_1 \div p_1$ leaves remainder 1; P_2 a multiple of π/p_2, so that $P_2 \div p_2$ leaves remainder 1, and so on. Let N divided by $p_1, p_2, p_3, \ldots$ leave remainders $r_1, r_2, r_3, \ldots$. Then $N - \Sigma P_1 r_1$ is a multiple of π. [The above is an extension of the theorem that, if N divided by p leaves remainder r, $N - r$ is divisible by p.]

Solution by H. W. CURJEL, B.A.

Evidently $N - P_1 r_1 \equiv r - P_1 r_1 \equiv 0 \pmod{p_1}$, and all the other terms of $N - \Sigma P_1 r_1$ are divisible by p_1; therefore $N - \Sigma P_1 r_1$ is divisible by p_1; similarly, it is divisible by p_2, p_3, &c., and is therefore a multiple of π.

From the statement of the question it is evident that $p_1, p_2, p_3, \ldots$ must be prime to one another.

12416. (Professor HEATON.)—Through three given points pass two spherical surfaces tangent to a given sphere.

Solution by Professors SCHOUTE, MOREL, *and others.*

The common tangential plane passes through the radical axis of the circle ABC (A, B, C being the given points) and the intersection of the plane ABC and the given sphere S, &c.

12425. (R. TUCKER, M.A.)—D, E, F are the mid-points of the sides BC, CA, AB of the triangle ABC; prove that, if $K \equiv a^2 + b^2 + c^2$, the "S"-points of AEF, BFD, CFE lie on the circle

$$4K^2 . \Sigma(a\beta\gamma) = \Sigma(a\alpha) . \Sigma\left\{bc\alpha\,(a^2 + 4b^2 + 4c^2)\right\}.$$

Solution by the PROPOSER; M. BRIERLEY; *and others.*

It is very readily seen that the trilinear coordinates of the "S"-point of AEF are proportional to $k + a^2$, ab, ac; with like values for the other two triangles. It is only necessary to verify the given equation by substitution.

9871. (B. F. FINKEL, B.Sc.)—Find the average distance from the centre of the base of all points of the arc of a parabola whose base is 40 inches and altitude 30 inches.

Solution by H. J. Woodall, A.R.C.S.

Take the centre of the base for origin of coordinates, and axis of parabola for axis of x; therefore equation is $y^2 = 400 - 40x/3$.

$$\text{Average distance} = \int_0^{30} \{1 + (dx/dy)^2\}^{\frac{1}{2}} (x^2 + y^2)^{\frac{1}{2}}\, dy \Big/ \int_0^{30} \{1 + (dx/dy)^2\}^{\frac{1}{2}},$$

$$x = 30 - 3y^2/40, \quad dx/dy = -3y/20.$$

Substituting, we get the required result.

12422. (Professor Sanjána, M.A. Suggested by Quest. 12027.)—The sides AB, AC of a triangle are produced to B″, C′, so that BB″ = CC′ = a; the sides BC, BA to C″, A′, so that CC″ = AA′ = b, and the sides CA, CB to A″, B′, so that AA″ = BB′ = c. Prove that, if α, β, γ stand for sin A, sin B, sin C, the area of A′A″B′B″C′C″ is

$$2R^2 \{\alpha(\alpha+\beta)(\alpha+\gamma) + \beta(\beta+\gamma)(\beta+\alpha) + \gamma(\gamma+\alpha)(\gamma+\beta) + \alpha\beta\gamma\}.$$

Solution by Professors Droz-Farny, Beyens, *and others*.

La figure tout entière se compose de quatre triangles égaux et de trois quadrilatères; on a

$$\Delta ABC + BCC'B'' = \tfrac{1}{2}a(a+b)(a+c) = 2R^2\alpha(\alpha+\beta)(\alpha+\gamma),$$
$$\Delta ABC + CAA'C'' = 2R^2\beta(\beta+\gamma)(\beta+\alpha),$$
$$\Delta ABC + ABB'A'' = 2R^2\gamma(\gamma+\alpha)(\gamma+\beta),$$
$$\Delta ABC = \tfrac{1}{2}abc = 2R^2\alpha\beta\gamma;$$

donc A′A″B′B″C′C″

$$= 2R^2 \{\alpha(\alpha+\beta)(\alpha+\gamma) + \beta(\beta+\gamma)(\beta+\alpha) + \gamma(\gamma+\alpha)(\gamma+\beta) + \alpha\beta\gamma\}.$$

9865. (Artemas Martin, LL.D.)—The following method has been given for "partial payments" when simple interest only is to be allowed: "The sum of the present worths of all the payments is equal to the debt." Show that this is not a *just* method, since it would enable the borrower to discharge his indebtedness by paying yearly a sum less than the *interest* of the debt; and find in what length of time a debt of \$100 could be cancelled by paying annually the sum of \$6, reckoning simple interest at 6 per cent. per annum.

Solution by H. J. Woodall, A.R.C.S.

According to the "rule," we have $D = \Sigma Ak/(1+kr)$, where D is the debt, Ak the sum paid at the end of the kth period, and r is the rate of interest. The defect of simple interest is that it does not allow for interest on unpaid interest, which is quite as evident here as elsewhere.

2. $100 = 6\Sigma\{1/(1+k \times \cdot 06)\}$; we find $n = 29\frac{1}{2}$ years nearly.

9522. (A. R. Johnson, M.A. Connected with Question 9480, Vol. LIII., p. 36.)—If $I = 0$ be the condition that a pair of lines should be equally inclined to the bisectors of the angles between the axes, and if R' be the resultant of $f(x, y)$ and $f(y, x)$, prove that

$$I = \left\{R'/f(1, 1)f(1, -1)\right\}^{\frac{1}{2}}.$$

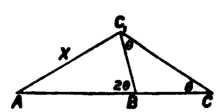
Solution by H. J. Woodall, A R.C.S.

Forming the resultant in the usual way, we shall have, since the roots of $f(x, y)$ are the reciprocals of the roots of $f(y, x)$,

$$\begin{aligned} R &= a_0^n a_n^n (\alpha - 1/\alpha)(\alpha - 1/\beta) \dots (\beta - 1/\alpha)(\beta - 1/\beta) \dots \\ &= a_0^n (\alpha^2 - 1)(\beta^2 - 1) \dots (\alpha\beta - 1)^2 \dots \\ &= a_0 (1-\alpha)(1-\beta) \dots a_0 (1+\alpha)(1+\beta) \dots a_0^{2n-2} I^2 \\ &= f(1, 1),\ f(1, -1),\ a_0^{2n-2} I^2, \text{ which gives the result.} \end{aligned}$$

12427. (H. D. Drury, M.A.)—Produce a line AB to C, so that AC, CB may be equal to the square on a given line X; then, if we construct a triangle whose sides are AB, BC, X, the angle opposite the side X will be double the angle opposite the side BC. Hence show how Prop. 10, Bk. IV., follows.

Solution by R. Chartres; Professor de Wachter; *and others.*



Given $BC = BC_1$;

since rec. $AC \cdot CB = X^2$,

therefore $CC_1 = X$, and $A = \theta$.

Hence, when $X = AB$, the triangle ABC_1 satisfies Euc. IV. 10.

12413. (Professor Morley, M.A.)—Ten men, sitting in a ring, place their hats within it. Then each man puts on one of the hats at random. What is the chance that no man has a neighbour's hat?

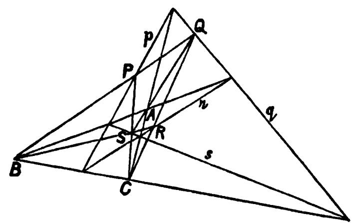
Solution by Professors Schoute, Mukhopadhyay, *and others.*

This question is not new: it was proposed by de Montmort, and solved by Euler. A solution by means of the chess-board is to be found in Lucas' *Théorie des Nombres*, Vol. I., p. 211 ("Problème des Rencontres").

If Q_n is the number of favourable permutations for n persons, the recurrent relation $$Q_n = nQ_{n-1} + (-1)^n$$ holds. So the number in question is $\frac{16481}{44800}$ or $0{\cdot}367\dot{8}5714\dot{2}$.

12622. (R. LACHLAN, Sc.D.)—A tetragram is constructed having fixed diagonals forming a triangle ABC. If one side of the tetragram pass through a fixed point P, show that each of the other sides will pass through a fixed point, and that, if these points be Q, R, S, then A, B, C will be the centres of the tetrastigm PQRS.

Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*



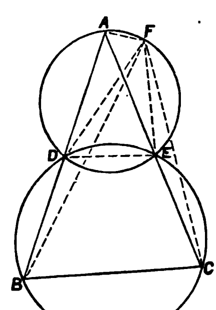
Let p, q, r, s be the sides of the tetragram, of which p passes through the fixed point P. Join P, A, cutting r in R. Then PR is cut harmonically at A and its point of intersection with BC; therefore R is a fixed point.

Similarly, s always passes through S, the harmonic conjugate of P with respect to C, and the intersection of AB with PC, and q through Q, the harmonic conjugate of P with respect to B, and the intersection of AC, PB. Similarly, QR, QS, SR pass through C, A, B, respectively; *i.e.*, A, B, C are the centres of the tetrastigm PQRS.

12545. (Professor SÉVOZ.)—Par les sommets B et C d'un triangle ABC, on fait passer un cercle qui coupe AB en D et AB en E; puis par les trois points A, D, E on mène un second cercle qui rencontre en F le cercle circonscrit au triangle ABC. Démontrer la relation

$$(FB+FE)/(FC+FD) = AB/AC.$$

Solution by Professors DROZ-FARNY, SANJÁNA, *and others.*



Les quadrilatères AFBC et AFDE étant inscriptibles, on a

$$AB \cdot FC + AF \cdot BC = FB \cdot AC \;\ldots\ldots\; (1),$$

$$FD \cdot AE = FE \cdot AD + FA \cdot DE \;\ldots\ldots\; (2).$$

Les triangles ADE et ABC sont semblables; donc $AD : AC = AE : AB = DE : BC$.

La relation (2) étant homogène par rapport aux segments AE, AD, DE, on peut les remplacer par leurs segments proportionnels; on a ainsi $FD \cdot AB = FE \cdot AC + FA \cdot BC \ldots (3)$, puis par addition des relations (1) and (2)

$$FB \cdot AC + FE \cdot AC = FC \cdot AB + FD \cdot AB,$$

$$AC(FB+FE) = AB(FC+FD),$$

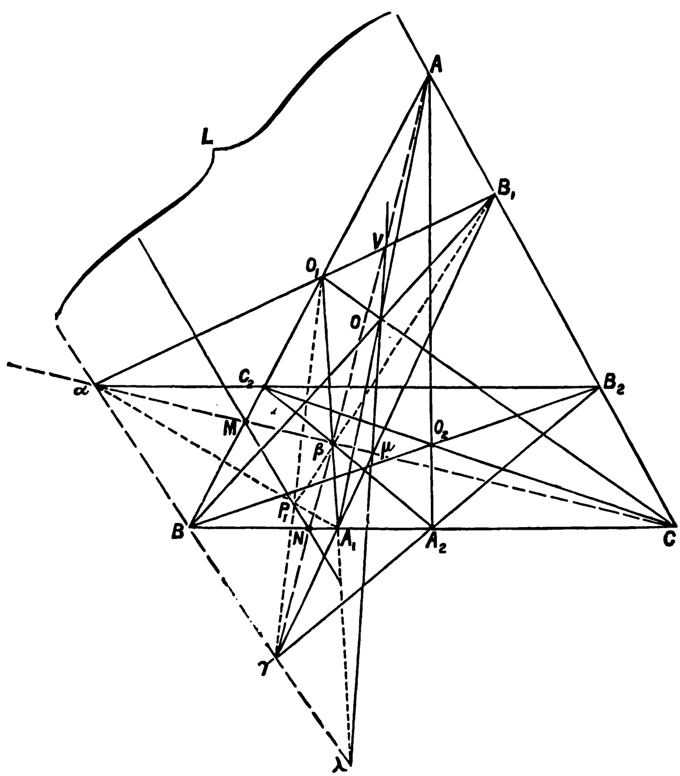
$$(FB+FE)/(FC+FD) = AB/AC.$$

[The three lines AF, DE, BC cut each other in the same point. This question is the same as Mr. TUCKER's Question 4630, whereof a Solution was published, many years ago, on page 77 of Vol. 23.]

12567. (R. LACHLAN, M.A.)—**Prove, geometrically, that if two triangles be in perspective with the triangle formed by the lines joining their vertices, the points of intersection of corresponding sides of the two triangles form a triangle in perspective with each of the given triangles and the triangle formed by the lines joining their vertices.**

Solution by Professors DROZ-FARNY, KRISHNAMACHARRY, *and others.*

Soient ABC le triangle donné, $A_1B_1C_1$ et $A_2B_2C_2$ deux triangles inscrits dans ABC et en perspective avec lui ; O_1 et O_2 sont les centres d'homologie. B_1C_1 et B_2C_2 se coupent en α, A_1C_1 et A_2C_2 en β et A_1B_1 et A_2B_2 en γ.



Les faisceaux A_1 (BC_1AB_1) et A_2 (BC_2AB_2) sont harmoniques ; comme

ils ont le rayon BC en commun, ils sont perspectifs; par conséquent les points $(A_1C_1, A_2C_2) \equiv \beta$, $(A_1A, A_2A) \equiv A$, $(A_1B_1, A_2B_2) \equiv \gamma$ sont en ligne droite; de même $B\alpha\gamma$ et $C\alpha\beta$ sont des lignes droites.

Dans l'hexagone dégénéré $C_1A_1B_1B\alpha C$ on a la pascale

$$(C_1A_1, B\alpha) \equiv \lambda, \quad (A_1B_1, C\alpha) \equiv \mu, \quad (B_1B, CC_1) \equiv O_1,$$

de même dans $A_1C_1B_1B\gamma A$ la pascale

$$(A_1C_1, B\gamma) \equiv \lambda, \quad (C_1B_1, \gamma A) \equiv \nu, \quad (B_1B, AA_1) \equiv O_1,$$

et dans $C_1B_1A_1A\beta C$ la pascale

$$(C_1B_1, A\beta) \equiv \nu, \quad (B_1A_1, \beta C) \equiv \mu, \quad (A_1A, CC_1) \equiv O_1.$$

Il en résulte que les points λ, μ, ν, O_1 sont en ligne droite et que les triangles $\alpha\beta\gamma$ et $A_1B_1C_1$ sont homologiques.

L'axe d'homologie est $\lambda\mu\nu$; représentons par P_1 le centre d'homologie point d'intersection des droites αA_1, βB_1, γC_1. Il suffit dans les hexagones précédents de remplacer A_1, B_1, C_1 par A_2, B_2, C_2; λ, μ, ν respectivement par λ', μ', ν', et on aura démontré que les triangles $\alpha\beta\gamma$ et $A_2B_2C_2$ sont homologiques; l'axe d'homologie est $\lambda'\mu'\nu'$ et le centre d'homologie point d'intersection des droites αA_2, βB_2, γC_2 sera représenté par P_2.

Dans l'hexagone dégénéré $\gamma\alpha\beta B_1AC_1$ on a la pascale

$$(\gamma\alpha, B_1A) \equiv L, \quad (\alpha\beta, AC_1) \equiv M, \quad (\beta B_1, \gamma C_1) \equiv P_1,$$

dans l'hexagone $\gamma\beta\alpha A_1BC_1$ on a la pascale

$$(\gamma\beta, A_1B) \equiv N, \quad (\beta\alpha, BC_1) \equiv M, \quad (\alpha A_1, C_1\gamma) \equiv P_1,$$

enfin dans $(\alpha\gamma\beta B_1CA_1)$ on a la pascale

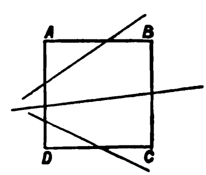
$$(\alpha\gamma, B_1C) \equiv L, \quad (\gamma\beta, CA_1) \equiv N, \quad (\beta B_1, \alpha A_1) \equiv P_1.$$

Il en résulte que les points L, M, N, P_1, P_2 sont en ligne droite et que les triangles $\alpha\beta\gamma$ et ABC sont homologiques, l'axe d'homologie étant LMN.

[Prof. Farny remarks that "de nombreuses propriétés de cette figure ont été indiquées par Prof. Schroeter dans les *Nouvelles Annales* et par Prof. Neuberg dans la *Nouvelle Correspondance de Mathématiques*."]

2901. (I. H. Turrell.)—A square is divided at random by two straight cuts. Required the probability that one of the pieces is a pentagon.

Solution by R. Chartres, Professor Aiyar, *and others*.

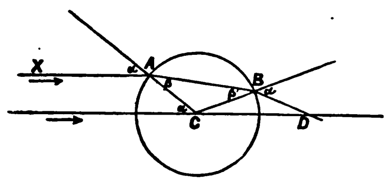


The first cut is equally likely to go through AB, BC, or CD, and similarly with the second cut, giving nine cases, of which six are favourable to one of the pieces being a pentagon;

hence the probability $= \frac{2}{3}$.

12608. (Professor LAMPE.)—Parallel rays of light passing through a transparent sphere are in general reflected in such a manner that only the rays with the same angle of incidence are united in a luminous point behind the sphere. If, however, the index of refraction of the sphere has a certain numerical value (following from the cubic $n^3-n^2-n-1=0$, say $n = 1{\cdot}839287$), the rays passing through the vicinity of the centre and those passing through the region most distant from the centre (or those corresponding respectively to the angles of incidence 90° and 0°) have the same point of union, lying $0{\cdot}09574r$ behind the sphere.

Solution by Professors LAMPE, BHATTACHARYA, *and others.*



Let XA be an incident ray; then we have

$$\angle BCD = 2\beta-\alpha,$$
$$\angle BDC = 2\alpha-2\beta;$$

therefore

$$\frac{CD}{BD} = \frac{\sin\alpha}{\sin(2\alpha-2\beta)};$$

or, putting $CD = d$ and making $\sin\beta = \sin\alpha/n$, n being the index of refraction, $d = n^2r/2\left\{\cos\alpha\,(n^2-\sin^2\alpha)-\cos 2\alpha\,(n^2-\sin^2\alpha)^{\frac{1}{2}}\right\}$.

For $\alpha = 0, = 90°$, this becomes

$$d_1 = \frac{nr}{2(n-1)}, \qquad d_2 = \frac{n^2r}{2(n^2-1)^{\frac{1}{2}}}.$$

Making $d_1 = d_2$, we get

$$n^3-n^2-n-1 = 0, \quad n = 1{\cdot}839287, \quad d_1 = d_2 = 1{\cdot}09574.$$

12614. (Professor NEUBERG.)—Intégrer l'équation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = \frac{2x+1}{x^2}e^{-x}.$$

Solution by Rev. J. L. KITCHIN, M.A.; H. W. CURJEL, M.A.; *and others.*

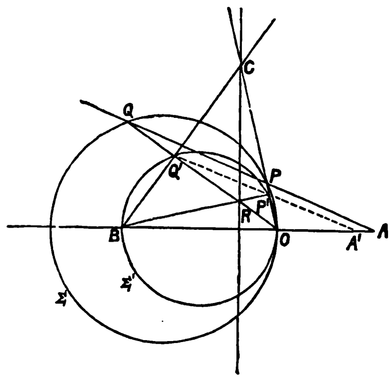
The equation may be written

$$\frac{d^2}{dx^2}(ye^x) = \frac{2x+1}{x^2}; \quad \text{therefore } ye^x = (2x-1)\log x + C_1x + C_2.$$

12560. (W. J. DOBBS.)—O is a fixed point on a circle; A and B fixed points on diameter through O; PQ a chord passing through A, and perpendicular from B on OP meets OQ in R. Find the locus of R.

Solution by Professor Droz-Farny; H. W. Curjel, M.A.; *and others.*

La perpendiculaire BP′ sur OP coupe OQ en R, et la perpendiculaire BQ′ sur OQ coupe OP en C. Les points P′ et Q′ sont sur la circonférence Σ′ décrite sur BO comme diamètre. O étant le centre de similitude directe des circonférences Σ et Σ′, la droite P′Q′ est parallèle à PQ et coupe le diamètre OA en un point fixe A′ tel que OA′; OA est égal au rapport des rayons des circonférences Σ′ et Σ. Dans le quadrilatère inscrit OBQ′P′, le lieu des points C et R est la polaire par rapport à Σ′ du point fixe A′.



12558. (D. Biddle.)—Describe a method of solving cubic equations by means of the slide rule, and of subsequently improving the result by means of logarithms.

Solution by the Proposer.

The slide rule may be used to solve the cubic equation $x^3+qx+r=0$, in which $1:x=x^2+q:-r$, by placing clips on the upper fixed portion to indicate the quantities q and r, and then moving the slide until 1 on it comes under (x), whilst r is over $(x^2)+q$. As a matter of constancy, (x) on the slide itself has (x^2) over it. The quantities represented by q and r can be equalized by taking $z = q/r\,x$, when we get

$$z^3+q^3/r^2\,z+q^3/r^2=0.$$

Then a clip is scarcely needed. For example,

	$6\ (=-r')$	$8{\cdot}08\ (=z^2)$	$2{\cdot}842\ (=z)$
$z^3-6z-6=0,$			
	$2{\cdot}08\ (=z^2+q')$	$2{\cdot}842\ (=z)$	1 on slide

A further approximating trial can now be made by logarithms. We know that $q+r/x : r+qx = x^2 : x^3$. Therefore take the logs of q, r, and (x) as found on the slide rule. Add $\log(x)$ to $\log q$, and subtract $\log(x)$ from $\log r$. Then the resulting numbers added to r and q respectively should be represented by logs having a ratio $3:2$. Alter $\log(x)$ until this is the case.

In order to do this expeditiously, take the logs of $q+r/(x)$ and $r+q\,(x)$, and, dividing the former by 2 and the latter by 3, observe the difference $(=d)$ between the quotients. This is the amount of error to be

eliminated. Let the respective increment or decrement, needed to render the quotients equal, be represented by u, v. Then $u+v=d$. Next, observe the numbers at the side of the logarithmic tables, indicating the differences in logs, at $q+r/(x)$ and $r+q\,(x)$, and let these respectively $=\delta$ and δ'. The corresponding differences of the numbers as compared with the logarithms will be in inverse ratio; therefore $2u : 3v = \delta' : \delta$,

whence $$u = \frac{3d\delta'}{2\delta+3\delta'}, \quad \text{and} \quad v = \frac{2d\delta}{2\delta+3\delta'}.$$

These must be severally added and subtracted so as to bring the original quotients into closer unison, and then the process be repeated if requisite. But one trial is generally sufficient, and it is easy to find x from the log of $r+qx$.

In the above example,

$$q'+r'/(z) = -8{\cdot}11119, \quad \text{and} \quad r'+q'\,(z) = -23{\cdot}052;$$

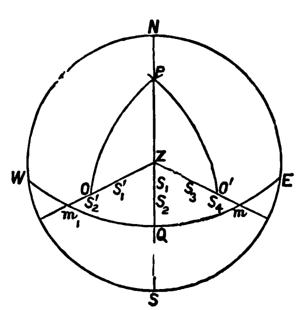
$$\delta = 54, \quad \delta' = 18{\cdot}9, \quad \tfrac{1}{2}\log\left\{q'+r'/(z)\right\} = 0{\cdot}4545423,$$

and $$\tfrac{1}{3}\log\left\{r'+q'\,(z)\right\} = 0{\cdot}4542362; \quad \therefore\ d = 3061,$$

whence $u = 1054$, $v = 2007$. Hence $\frac{1}{3}\log\left\{r'+q'z\right\} = 0{\cdot}4544369$. This gives $z = 2{\cdot}8473$, which is very nearly correct.

12571. (Professor Haughton, F.R.S.)—In crossing a river I have lost my sextant, which slipped off my horse's back and disappeared; fortunately my watch is safe, and my papers, including a table of declination and right ascension of the principal stars. How shall I find my latitude by noting the times at which any two pairs of stars are on the same vertical?

Solution by C. Bickerdike; Professor Bhattacharya; *and others.*



Let S_1, S_2 represent the position of the pair of stars on the observer's vertical at the time t; and S_3, S_4 the position of the second pair at the same time on the vertical ZO′. Then when the stars S_3, S_4 have moved from ZO′ to ZQ, the stars S_1, S_2 will have moved from ZQ to ZO (say) in the time t_1; hence t_1-t is angle OPZ.

Hence, in the triangle OPZ, we have given OP, codeclination (from tables); angle OPZ, the observer's time converted in arcual measure; and the angle OZP a right angle, to find the colatitude ZP.

12621. (EDITOR.)—In the circumference of a circle find a point such that the lines that join it to two given points in the circumference shall intercept, on a given chord of the circle, a given length.

Solution by Professors DROZ-FARNY, AIYAR, *and others.*

Soient L, M, N, ... et L′, M′, N′, ... 2 ponctuelles homographiques sur une droite donnée; construisons les points λ, μ, ν, ... de manière à ce que $L\lambda = M\mu = N\nu =$ une longueur déterminée. Les ponctuelles L, M, N, ... et λ, μ, ν, ... étant égales, les ponctuelles λ, μ, ν, et L′, M′, N′ seront homographiques et admettent 2 points doubles qui seront par conséquent à distance donnée de leurs homologues dans la série L.

Si donc 2 ponctuelles homographiques sont sur un même support, il existe deux segments entre points homologues de longueur donnée.

Soient A et B deux points fixes d'une circonférence et C un point variable; les faisceaux AC et BC sont homographiques et coupent par conséquent sur une corde fixe des ponctuelles α et β homographiques. Il suffira donc de chercher d'après la méthode précédente les segments homologues de longueur donnée. Le problème admet deux solutions.

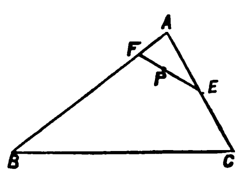
12590. (R. TUCKER, M.A.)—FE is a positive-oblique isoscelian (to A) cutting AB in F, and AC in E; on FE a point P is taken so that $FP : PE = m : n$. Points Q, R are similarly taken on the P.O.I. to B and C; show that, if $m^3 : n^3 = 8 \cos A \cos B \cos C : 1$,

then AP, BQ, CR meet in T, given by (in trilinears)

$$(\cos C)^{\frac{2}{3}} (\cos A)^{\frac{1}{3}} : (\cos A)^{\frac{2}{3}} (\cos B)^{\frac{1}{3}} : (\cos B)^{\frac{2}{3}} (\cos C)^{\frac{1}{3}}.$$

Also, if the same be done for the negative-O.I. (F′E′, &c.), then the point T′ is the isogonal conjugate of T.

Solution by Professor SANJÁNA, PROPOSER, *and others.*



The equations to AP, BQ, CR are

$$\frac{\beta}{2n \cos A} = \frac{\gamma}{m}, \quad \frac{\gamma}{2n \cos B} = \frac{\alpha}{m}, \quad \frac{\alpha}{2n \cos C} = \frac{\beta}{m},$$

whence $\quad m^3 = 8n^3 \cos A \cos B \cos C \; \ldots\ldots \; (1),$

and the point T is given by

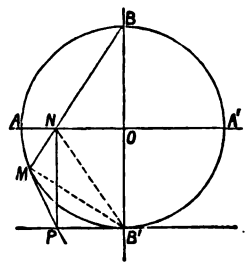
$$\frac{\alpha}{m^2} = \frac{\beta}{4n^2 \cos A \cos B} = \frac{\gamma}{2mn \cos B}.$$

Substituting from (1), we get the result required.

The like result for T′ is obtained in the same way.

12618. (Professor LAPERROUSAZ.)—On considère dans un cercle deux diamètres rectangulaires AA′ et BB′. Par l'extrémité B du diamètre BB′, on mène une sécante quelconque qui rencontre le cercle en M et le diamètre AA′ en N. Trouver le lieu du point d'intersection de la tangente en M au cercle et de la perpendiculaire élevée en N à AA′.

Solution by Professor DROZ-FARNY; Rev. J. L. KITCHIN, M.A.; *and others.*



Soit P le point de coupe des droites considérées. On a

$$\angle \text{PMB}' = \text{B} = \text{PNB}'.$$

Le quadrilatère NMPB′ est donc inscriptible, et comme angle NMB′ = 90°, NB′ est un diamètre; donc angle NPB′ est aussi droit et par conséquent aussi angle PB′O. Le lieu cherché est la tangente en B′ au cercle.

12630. (J. H. HOOKER, M.A.)—Find two squares such that their sum is the double of a square, and their difference is ten times a square.

Solution by Major-General O'CONNELL; *and* Rev. J. L. KITCHIN, M.A.

Putting $\qquad x = p^2 - q^2 + 2pq, \quad y = -p^2 + q^2 + 2pq,$

we have $\qquad x^2 + y^2 = 2(p^2+q^2)^2,$

which satisfies the first condition. From these values, $x^2 - y^2 = 8pq(p^2-q^2)$; hence, assuming $p = 5m$, $q = 4m$, we have $8pq(p^2-q^2) = 10(12m^2)^2$, the second condition. Hence $x = 49m^2$, $y = 31m^2$, where m has any value, integral or fractional.

[Mr. DAVIS gives the solution thus:—The first condition is satisfied by the assumption that the two numbers required are of the respective forms $n^2 + 2n - 1$, $-n^2 + 2n + 1$; and the second condition requires that $8(n^3 - n)$ should be $= 10\,\square$; or that $n^3 - n = 5\,\square$, which is obviously satisfied by the value $n = 9$. The numbers required are therefore 98, 62.]

12523. (W. J. DOBBS, M.A.)—In getting up speed, assume that a steamer moves so that its acceleration at any instant varies as the defect of its velocity from its maximum velocity. Supposing it to describe a straight course, while the wind blows steadily across its path at right angles, find the equation to the line of trail of its smoke at a given time after the start. Show that the line of trail is an orthogonal projection of $x + y + 1 = e^y$.

Solution by H. W. Curjel, M.A.; Professor Aiyar; *and others*.

Let V be the maximum velocity and s the space described in the time t.

Then $\ddot{s} = a(\mathrm{V} - \dot{s})$; therefore $\dot{s} = \mathrm{V}(1 - e^{-at})$;

therefore $$s = \mathrm{V}t + \mathrm{V}\frac{e^{-at} - 1}{a};$$

therefore the equation to the trail at any time t is given by eliminating t_1 between

$$-\left\{x - \frac{\mathrm{V}(at_1 + e^{-at_1} - 1)}{a}\right\} = \frac{\mathrm{V}}{a}(at + e^{-at} - 1), \quad y - \mathrm{U}t_1 = -\mathrm{U}t,$$

where U is the velocity of the wind. This curve is an orthogonal projection of $x + y + 1 = e^y$.

9224. (F. Morley, B.A.)—Find $\int_0^{2-\sqrt{2}} \log\frac{1-x}{1-\frac{1}{2}x}\frac{dx}{x}$.

Solution by H. J. Woodall, A.R.C.S.; Professor Aiyar; *and others*.

We get, by expansion,

$$\text{Integral} = -\left\{\tfrac{1}{2}x + \tfrac{3}{16}x^2 + \tfrac{7}{72}x^3 + \tfrac{15}{256}x^4 + \tfrac{31}{800}x^5 + \frac{2^n - 1}{n^2 \cdot 2^n}x^n\right\}.$$

Substituting $x = 2 - \sqrt{2}$, we get the required value $-\cdot3880$.

12595. (I. Arnold.)—There is an inclined plane 500 feet long and 300 feet high. A heavy weight slides freely down the plane. At what distance from the top of the plane will it begin to describe a space equal to the height, and in the same time that it would have fallen through the height by the force of gravity?

Solution by Beatrice A. Ward, B.Sc.; Professor Chakrivarti; *and others*.

Height of plane is 300 ft.; hence the time taken to fall the height $= \frac{5}{4}\sqrt{3}$ secs. The weight would describe space equal to height (300 ft.) in time taken to fall the height ($\frac{5}{4}\sqrt{3}$ secs.), if initial velocity, when it commences to describe the space, is equal to $16\sqrt{3}$ ft. per sec. (Given by equation $s = ut + t^2 g \,\frac{1}{2}\sin\theta$.) And distance (D) from top of plane in which this velocity would be acquired is 20 ft. Hence the required distance is 20 ft.

12537. (Professor Orchard, B.Sc., M.A.)—The elliptic lamina $7x^2 + 9y^2 = 28$ has two equal particles placed at the ends of its minor axis, and the tangent and normal at any point are the principal axes there. Show that the mass of each particle is one-fourth that of the disc.

Solution by Rev. D. T. Griffiths, M.A.; Prof. Sarkar; *and others*.

The tangent and normal at any point are the principal axes there, provided foci of inertia coincide with foci of ellipse.

Required condition, adopting usual notation,

$$ae = \left[(A-B)/(M+2m)\right]^{\frac{1}{2}}, \quad A = \tfrac{1}{4}Mb^2 + 2mb^2, \quad B = \tfrac{1}{4}Ma^2;$$

$$\therefore \quad m/M = \tfrac{1}{8} \, . \, e^2/(1-2e^2).$$

[This is the solution of an example given in Routh's "Rigid Dynamics."]

For the particular case in question, put $e^2 = \frac{1}{3}$, and we have $m = \frac{1}{8}M$.

12460. (J. W. Russell, M.A.)—Show that, if the reciprocal of the conic S with respect to the conic S′ coincides with S, then the reciprocal of S′ with respect to S coincides with S′; and find the relations which hold in this case between the invariants Δ, Θ, Θ', and Δ'.

Solution by the Proposer.

If S is $\Sigma l\alpha^2 = 0$, and S′ is $\Sigma u\alpha^2 = 0$, then the reciprocal of S′ with respect to S is $\Sigma u^2\alpha^2/l = 0$. And this coincides with $\Sigma l\alpha^2 = 0$;

$$\therefore \quad u = \pm l, \quad v = \pm m, \quad w = \pm n.$$

Hence S′ is $l\alpha^2 \pm m\beta^2 \pm n\gamma^2 = 0$. This proves the first part.

Now form the discriminant of $l\alpha^2 + m\beta^2 + n\gamma^2 + k\,(l\alpha^2 + m\beta^2 - n\gamma^2) = 0$. It is $k^3 + k^2 - k - 1 = 0$; $\quad \therefore \quad \Theta^2 + \Delta\Theta' = 0$ and $\Theta'^2 + \Delta'\Theta = 0$.

12568. (F. G. Taylor, M.A., B.Sc.)—If

$$u_n + \frac{n}{1} u_{n-1} + \frac{n(n-1)}{1 \, . \, 2} u_{n-2} + \ldots = v_n,$$

prove that
$$v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1 \, . \, 2} v_{n-2} - \ldots = u_n.$$

Solution by H. W. Curjel, M.A.; Professor Sarkar; *and others*.

$$(E+1)^n u_0 = v_n;$$

$$\therefore \quad u_n = E^n u_0 = \left\{(E+1)-1\right\}^n u_0$$

$$= (E+1)^n u_0 - \frac{n}{1}(E+1)^{n-1} u_0 + \frac{n(n-1)}{1 \, . \, 2}(E+1)^{n-2} u_0 - \ldots$$

$$= v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1 \, . \, 2} v_{n-2} - \ldots.$$

12510. (Professor Hudson, M.A.)—When wheat is 1s. a bushel, bread is $\frac{1}{4}$d. a lb. What should the 4-lb. loaf cost when wheat is 28s. a quarter?

Solution by T. Savage; B. T. Brierley; *and others.*

At 8s. a quarter, the 4 lbs. cost 1d.; hence required price is $3\frac{1}{2}$d.

12577. (Professor Finkel.)—
Between Sing-Sing and Tarry-Town, I met my worthy friend, John Brown,
And seven daughters, riding nags, and every one had seven bags;
In every bag were thirty cats, and every cat had forty rats,
Besides a brood of fifty kittens. All *but* the nags were wearing mittens!
Mittens, kittens—cats, rats—bags, nags—Browns,
How many were met between the towns?

Solution by H. J. Woodall, A.R.C.S.

There were 8 Browns, 8 nags, 56 bags, 1,680 cats, 67,200 rats, 84,000 kittens, 611,552 mittens. Altogether, 764,504.

12572. (Professor Hudson, M.A.)—If 80 million equal and similar rectangles placed end to end extend 15,000 miles, and cover 1,300 acres, find the dimensions of one of the rectangles to the nearest half-inch. If their aggregate weight is 290 tons, find the weight of each.

Solution by H. J. Woodall, A.R.C.S.; Professor Aiyar; *and others.*

Length of one = 15,000 miles divided by 80 million = 12 inches, nearly.
Breadth ,, = 1,300 acres divided by 15,000 miles = $8\frac{1}{2}$,, ,,
Weight ,, = 290 tons divided by 80 million = $56\frac{1}{2}$ grains, ,,

3235. (Professor Hudson, M.A.)—Find the shape of a uniform wire such that the moment of inertia of any portion of it bounded by two radii vectors about an axis through the pole perpendicular to its plane may vary as the angle between them.

Solution by H. W. CURJEL, M.A.

The differential equation to the curve is evidently

$$r^2\, ds/d\theta = \text{a constant} = a^3, \text{ say};$$

therefore $r^2\left\{(dr/d\theta)^2 + r^2\right\}^{\frac{1}{2}} = a^3$; therefore $r^2 dr/(r^6 - a^6)^{\frac{1}{2}} = d\theta$;

therefore $$r^3 = a^3 \sin(3\theta + \alpha).$$

The equation is evidently also satisfied by $r = a$.

12384. (W. J. GREENSTREET, M.A.)—Given a conic S and two fixed points A, B. Show that the locus of the vertices of conics passing through A, B, axes parallel and proportional to those of S, breaks up into two conics, the one with its centre at the origin, the tangents at the given points being parallel to Oy; the other having its centre at the origin and the tangent at the given points parallel to Ox.

Solution by the PROPOSER; Professor SANJÁNA; *and others.*

Let a, b, (α, β) be respectively the semiaxes and centre of the given conic. Then, if we take as axes of the system in the problem the parallels to the axes of the given conic drawn through the mid-point A, B, these points will be (m, n), $(-m, -n)$.

The given conic will be $(x-\alpha)^2/a^2 + (y-\beta)^2/b^2 - \lambda = 0$, where

$$(m-\alpha)^2/a^2 + (n-\beta)^2/b^2 - \lambda = 0 = (m+\alpha)^2/a^2 + (n+\beta)^2/b^2 - \lambda.$$

Adding and subtracting these two relations,

$$m^2/a^2 + n^2/b^2 + \alpha^2/a^2 + \beta^2/b^2 = 0, \quad \text{and} \quad \alpha m/a^2 + \beta n/b^2 = 0 \;\ldots\ldots\; (1).$$

The vertices on the axis parallel to Ox satisfy the conditions

$$y = \beta, \quad (x-\alpha)^2 = a^2\lambda;$$

whence $$\alpha = -\frac{ayn}{b^2 m}; \quad \lambda = \left(\frac{x}{a} + \frac{ayn}{b^2 m}\right)^2;$$

$$\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\left(1 + \frac{a^2 y^2}{b^2 m^2}\right) = \left(\frac{x}{a} + \frac{any}{b^2 m}\right)^2;$$

$$\frac{x^2}{a^2} + \frac{2xyn}{b^2 m} - \frac{y^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2} = k^2 \text{ (say)},$$

where ka, kb are the semiaxes of a conic homothetic to S, passing through A, B, and having the centre as origin.

Hence the locus required will be a conic S′, diameter parallel to Ox being $2ka$, passing through A, B, tangents at A, B parallel to Oy, centre the origin. The locus of the vertices on the axis parallel to Oy is

$$+\frac{x^2}{a^2} - \frac{2xym}{a^2 n} - \frac{y^2}{b^2} = -k^2,$$

a conic of the same nature as S′, cutting the axis of y at points $y = \pm kb$, tangential at A, B to the parallels drawn through them to Ox.

S′ is an hyperbola when S is an ellipse.

9752. (Professor MADHAVARAO.)—If G be the centre of inertia of n particles of masses $m_1, m_2, m_3, \dots m_n$, placed at the points $A_1, A_2, A_3, \dots A_n$, taken anywhere in space, and if from any point O $OA_1, OA_2, \dots OA_n$ be drawn and produced to $a_1, a_2, \dots a_n$, so that $Oa_1 = m_1 . OA_1$, $Oa_2 = m_2 . OA_2 \dots Oa_n = m_n . OA_n$, and if the circumference of a circle of radius $(m_1 + m_2 + \dots + m_n)\, OG/n$, passing through O and having its centre in OG, be divided into n equal parts at O, P, Q, R..., show that the forces represented in magnitude and sense by the n right lines formed by joining one of the points $a_1, a_2, \dots a_n$ with one of the points O, P, Q, R, ..., another of the former with another of the latter, and so on, are in equilibrium.

Solution by H. J. WOODALL, A.R.C.S.

Let C be the centre of the circle; therefore CO = radius. Denote a system by Σ; then $\Sigma RC = 0 = \Sigma m_k A_k G$ (*i.e.*, they are in equilibrium),

$$\begin{aligned}\Sigma Ra_k &= \Sigma RC + \Sigma CO + \Sigma Oa_k = nCO + \Sigma m_k OA_k \\ &= -nOC + \Sigma m_k OG + \Sigma m_k GA_k = -OG\,\Sigma m_k + OG\,\Sigma m_k = 0.\end{aligned}$$

Therefore the given system of forces is in equilibrium. The solution suggests that the theorem may be extended thus:—Let C be C.G. of n equal masses O, P, Q, R, ...; nCO be equal $(m_1 + m_2 + \dots + m_n)$ OG, all other points as in the given theorem; then the system of forces represented by Ka_k is in equilibrium.

12295. (Professor HUDSON, M.A.)—If the expenses of a scholastic agency are £250 a year, and this is recouped by a payment of $1\frac{1}{2}$ per cent. on the salaries of those who receive appointments, and all but 11·1 per cent. of the applicants receive appointments of the average value of £150 a year, find how many applicants the agency must have on its books in order to pay expenses.

Solution by J. C. STRACHAN; H. W. CURJEL, B.A.; *and others.*

Let x = number of applicants. Income of agency $= x \times \frac{8}{9} \times 150 \times 1{\cdot}5/100$. This must = £250 to pay expenses; hence $x = 125$.

11983. (W. J. GREENSTREET, M.A.)—Find the locus of the centres of conics (1) passing through a fixed point, touching a given straight line in a given point, and having $a^2 + b^2 = k^2$; (2) of constant area, passing through two fixed points and touching a fixed straight line; (3) of constant area, passing through a fixed point and touching Ox, Oy; (4) of constant area, passing through two fixed points and touching two given straight lines.

Solution by H. J. WOODALL, A.R.C.S.; Prof. BHATTACHARYA; *and others.*

(1) Conic touching Ox at (o, o) is $ax^2+2hxy+by^2+2fy = 0$; this passes through (l, m) if

$$al^2+2hlm+bm^2+2fm = 0 \qquad\text{.......................(1)};$$

centre is given by $\quad ax+hy = 0, \quad hx+by+f = 0$ ....(2, 3),
i.e., centre is $\quad \left\{hf/(ab-h^2), \; -af/(ab-h^2)\right\}$.
Transforming to this centre as origin and with parallel axes, we get equation $\quad ax^2+2hxy+by^2 = af^2/(ab-h^2)$,

and we find sum of squares of semiaxes $= (a+b)\,af^2/(ab-h^2)^2 = k^2$ ...(4).
To find required locus we must eliminate a, h, b, f between (1), (2), (3), (4). After some work, we get

$$(xm-yl)^4 = 0, \quad\text{or}\quad y\,(ym^2-2my^2+2xlm-yl^2-2mx^2) = k^2m\,(m-2y).$$

(2) Let the line be $y = k$, and the points (l, m), $(-l, -m)$, that is to say, one axis is parallel to the given line, and in such position that the origin is half way between the given points.

Then $ax^2+2hxy+by^2+2gx+2fy+c = 0$ touches the line if

$$k^2\,(h^2-ab)+2k\,(hg-af)+g^2-ac = 0 \qquad\text{..................(1)},$$

and passes through the points if

$$al^2+2hlm+bm^2+c = 0, \quad gl+fm = 0 \qquad\text{...............(2, 3)}.$$

If area of conic $= p$, then $\; p^2 = \pi^2\Delta^2/(ab-h^2)^3$ ....(4).
Put $p = \pi P$, where $\Delta = abc+2fgh-af^2-bg^2-ch^2$, and the centre is given by $\quad ax+hy+g = 0, \quad hx+by+f = 0$ ....(5, 6).
From these six equations we have to eliminate a, h, b, g, f, c. We find

$$P^2\,(ly-mx)^2\,(y^2-m^2)^2\,(k-y)^{-4}$$
$$= \left[2lx\,(xy+lm)-\left\{P^2\,(k-y)^{-2}+x^2+l^2\right\}(ly+mx)\right](ly-mx)(y^2-m^2)$$
$$-\left[\left\{P^2\,(k-y)^{-2}+x^2+l^2\right\}my-lx\,(y^2+m^2)\right]^2.$$

(3) Let $ax^2+2hxy+by^2+2gx+2fy+c = 0$ be the conic, then, since it touches both axes, $\quad g^2 = ac, \quad f^2 = bc$ ....(1, 2),
and passes through a given point (l, m) if

$$al^2+2hlm+bm^2+2gl+2fm+c = 0 \qquad\text{.....................(3)}.$$

Since it has constant area, therefore $P^2 = \Delta^2/(ab-h^2)^3$ ....(4);
the equations to its centre are

$$ax+hy+g = 0, \quad hx+by+f = 0 \qquad\text{.....................(5, 6)}.$$

From these six equations, we have to eliminate a, h, b, g, f, c. We find $\;(ly-mx)^6\left[P^2-y^2x^2+\left\{(y-m)(l-x)\pm(m^2-2y)^{\frac{1}{2}}(l^2-2x)^{\frac{1}{2}}\right\}^2\right] = 0.$

6520. (R. TUCKER, M.A.)—A cube is cut into two parts by a plane the trace upon which is a regular hexagon; find position of centre of inertia of one of the parts.

Solution by H. J. Woodall, A.R.C.S.

Let the length of an edge be a. The plane of section cuts each of the bounding squares in the mid-points of two adjacent edges. Hence the areas of each square face are divided thus into parts equal to $\frac{7}{8}a^2$ and $\frac{1}{8}a^2$, opposite parallel faces having opposite corners cut off. Hence one of the two solids has its base $= \frac{7}{8}a^2$ and its top $= \frac{1}{8}a^2$. The area of any parallel section plane, at height $= x$, may be put $= \frac{7}{8}a^2 - \frac{3}{4}ax$.

The height of C.G. is thus found to be

$$\int_0^a \left(\tfrac{7}{8}a^2 - \tfrac{3}{4}ax\right) x\,dx \bigg/ \int^a \left(\tfrac{7}{8}a^2 - \tfrac{3}{4}ax\right) dx = \tfrac{3}{8}a.$$

Thus the C.G. of such solid is at a point which is distant $\frac{3}{8}a$ from each of the faces comprising the angle furthest from the plane of section.

6593. (C. Leudesdorf, M.A.)—Defining a self-conjugate pentagon with regard to a conic as being such that each of its vertices is the pole of the side opposite to it, show that, if two conics $S = 0$, $S' = 0$ are such that pentagons self-conjugate with regard to the latter can be inscribed in the former, then $\Theta'(\Theta'^2 - 4\Delta'\Theta) + 4\Delta\Delta'^2 = 0$.

Solution by H. J. Woodall, A.R.C.S.

Take for S the circum-conic $\equiv lyz + mzx + nxy = 0$,

and $$S' \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Let AB′CDE be one of the pentagons above defined. The polar of $(x_1y_1z_1)$ with respect to $S' = 0$, is

$$x(ax_1 + gz_1 + hy_1) + y(by_1 + fz_1 + hx_1) + z(cz_1 + fy_1 + gx_1) = 0.$$

Polar of A $(1, 0, 0)$ is $ax + hy + gz = 0$; this passes through C $(0, 0, 1)$; therefore $g = 0$. Similarly polar of C passes through A. D is at the intersection of $ax + hy = 0$ and $S = 0$; whence D is

$$\left\{(mh - la)\,z/na,\ \ (la - mh)\,z/nh,\ \ z\right\}.$$

Then we find the polar of D (which is AB′), viz.,

$$y\left\{(la - mh)(ab - h^2) + nahf\right\} + za\left\{nch - mfh + laf\right\} = 0.$$

So, again, polar of C is $fy + cz = 0$; this cuts S in E (and A); polar of E is CB′, viz., $cx(la - mfh + nch) + y\left\{(nc - mf)(bc - f^2) + hcfl\right\} = 0$.

The condition that AB′, CB′ meet on S is

$$\Theta'(\Theta'^2 - 4\Delta'\Theta) + 4\Delta\Delta'^2 = 0,$$

where $$\Delta = 2lmn, \quad \Delta' = abc - af^2 - ch^2,$$

$$\Theta = -l^2a - m^2b - n^2c + 2mnf + 2lmh, \quad \Theta' = -2(laf - mhf + nch),$$

the invariants of the system.

12210. (H. J. WOODALL, A.R.C.S.)—Show that

$$\cot^{-1} 68 = \cot^{-1} 239 + 2\cot^{-1} 268 + \cot^{-1} 327.$$

Solution by R. CHARTRES.

The formula $(a^2+1)(2a+x_1+x_2+x_3) - x_1x_2x_3 = 0$, given by Mr. WOODALL on page 72 of Vol. LIX., is satisfied by

$$a = 68, \quad x_1 = (239-68), \quad x_2 = (327-68), \quad x_3 = \left(\frac{269 \times 267}{2 \times 268} - 68\right);$$

$$\therefore \quad \cot^{-1} 68 = \cot^{-1} 239 + \cot^{-1} 327 + 2\cot^{-1} 268.$$

12177. (S. TEBAY, B.A.)—If a, b, c are conterminous edges of a tetrahedron; α, β, γ the angles contained by bc, ca, ab; A, B, C the dihedral angles through a, b, c; A_1, A_2, A_3 the areas of the faces contained by bc, ca, ab; and V the volume of the tetrahedron; prove that

$$\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma} = \frac{3}{4} \cdot \frac{abc}{A_1A_2A_3} V.$$

Solution by Professors DROZ-FARNY, SARKAR, *and others.*

On sait que $\dfrac{\sin A}{\sin \alpha} = \dfrac{\Delta}{\sin \alpha \sin \beta \sin \gamma}$ où Δ est le sinus du trièdre. On sait aussi que $V = \frac{1}{6}abc\Delta$. Donc

$$\frac{\Delta}{\sin \alpha \sin \beta \sin \gamma} = \frac{6Vabc}{a^2b^2c^2 \sin \alpha \sin \beta \sin \gamma} = \frac{6Vabc}{(ab \sin \gamma)(ac \sin \beta)(bc \sin \alpha)}$$

$$= \frac{3Vabc}{4A_1A_2A_3}.$$

9732. (Professor CANTOR.)—Un trièdre trirectangle OXYZ étant coupé par un plan quelconque suivant le triangle ABC, démontrer la relation.

$$\cot A : \cot B : \cot C = OA^2 : OB^2 : OC^2.$$

Solution by H. W. CURJEL, B.A.

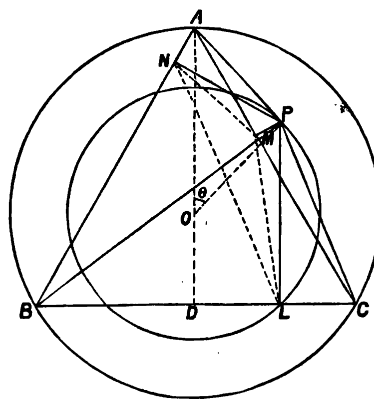
Using the usual notation for the triangle ABC,

$$\frac{\cot A}{OA^2} = \frac{b^2+c^2-a^2}{4S \cdot OA^2} = \frac{OC^2+OA^2+OB^2+OA^2-OB^2-OC^2}{4S \cdot OA^2} = \frac{1}{S2}$$

therefore the theorem follows.

9675. (Professor Darboux.)—Étant donnés un triangle équilatéral ABC et une circonférence concentrique Δ, les triangles qui ont pour sommets les projections d'un point quelconque de Δ sur les côtés de ABC ont même angle de Brocard.

Solution by R. Tucker, M.A.



$$MN = AP \sin 60°,$$
$$NL = PB \sin 60°,$$
$$LM = PC \sin 60°;$$

hence all triangles LMN are similar to triangles with sides PA, PB, PC.

$$\Sigma(PA^2) = 3(R^2+r^2) - 2Rr\left\{\cos\theta + \cos(120°+\theta) + \cos(120°-\theta)\right\}$$
$$= 3(R^2+r^2);$$

$$\Sigma(PA \, . \, PB)^2 = 3(R^2+r^2)^2 - 2Rr(R^2+r^2)(0) + 2R^2r^2\left(-\tfrac{3}{2}\right);$$

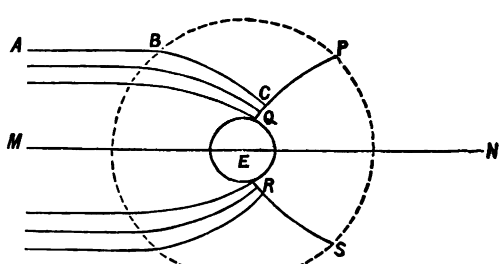
therefore K' and Δ' are constant; hence, &c.

[For a Solution by Mr. Woodall, see Vol. LXII., pp. 65, 66.]

12594. (S. Tebay, B.A.)—Give a simple but general explanation, with diagrams, of the Zodiacal Light and the Cometary Theory.

Solution by the Proposer.

Let E be the centre of the earth, supposed spherical, MN a line through



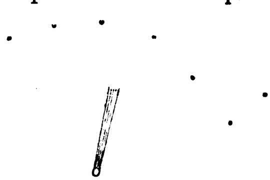
E parallel to the incident rays. Where the air begins to be sensible, as

at B, a ray AB will begin to bend, as BC, towards the line MN, in passing from rarer into denser medium. A similar course will be pursued by all rays falling on the earth's atmosphere parallel to MN, but their directions during refraction will not be parallel their successive intersections forming a caustic surface of revolution, in form of a cup, a section of which is shown as PQRS, and will appear, to a person situated in a suitable position on the earth, within the cup in the form of a spade. The height of the atmosphere being small compared with the radius of the earth, the elevation of the light will be still less, and will vary in appearance according to the position of the earth in its orbit. It will be most conspicuous between the tropics, or near the equator, the air being more rarefied and less absorbent of the sun's rays. This optical appearance is the phenomenon of the Zodiacal Light.

As an extension of this principle, suppose the nucleus of a comet to be surrounded by a thin vapour of extreme tenuity. When at a great distance from the sun, it will appear as a mere solid on account of extreme cold; but, as it nears the sun, its atmosphere will be expanded by the heat of the sun, and, during its perihelion passage, the caustic cup (as in the Zodiacal Light) will appear as an elongated tail. Were the comet at rest, the tail would be straight; but the rays of light in passing in front of the comet will suffer a slight deviation from this course on meeting with denser medium presented by the motion of the comet, and will fall below the line MN. Whence the sword-like form of the tail.

The following sketches were taken from observation of DONATI's comet of 1858 during its perihelion passage:—

September 26th. 7.30 p.m.

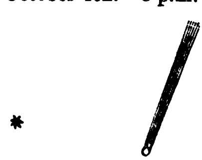


September 30th. 7 p.m.

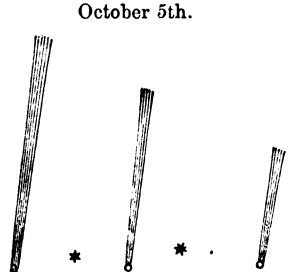
October 1st, 2nd, 3rd. Cloudy.

October 4th. 8 p.m.

Arcturus



October 5th.



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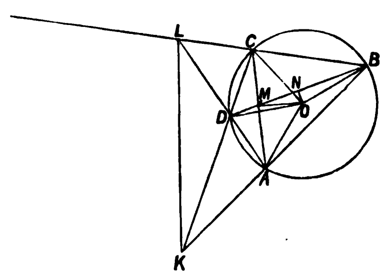
Figures 1, 2, 3 show the position of the comet and the star Arcturus at 6.15, 7.15, 8.15.

[The PROPOSER adds that, for some time past, he has thought that these subjects might admit of a mathematical explanation rather than a philosophical hypothesis. The phenomenon of the comet's tail has always been a mystery. The late Sir WILLIAM HERSCHELL suggested that there might be matter which admitted of repulsion as well as matter following the Newtonian law. But this hypothesis, being antagonistic to the established law of nature, has met with few supporters. The true explanation seems to be purely optical. The various phases assumed by HALLEY's comet, during its perihelion passage in 1835, are here sufficiently accounted for. It is recorded that the comet of 1744 had six tails, spread out like an immense fan. As regards our conception of a si gle nucleus, this would seem to be an incongruous anomaly. But if we consider it to have been a cluster of six comets, these, during their perihelion passage in a very eccentric orbit, might be sufficiently separated that each would exhibit its own individual tail.]

9118. (EDITOR.) — In a circle inscribe a complete quadrilateral, of which the three diagonals are given.

Solution by H. J. WOODALL, A R.C.S.



The two shorter diagonals will be chords of the given circle, and will therefore touch certain concentric circles respectively. If one of these chords be given in position, the other will be found when the angle between them is known. The problem, then, is to find the angle between the perpendiculars to these two chords (*i.e.*, between OM and ON).

Let $\quad OA = r, \quad AC = a = 2r \sin \alpha, \quad BD = b = 2r \sin \beta,$

$$MON = \theta, \quad AOM = \alpha = MOC, \quad NOB = \beta = NOD.$$

We find that

$$OAB = OBA = \tfrac{1}{2}\{-\pi + (\beta + \theta + \alpha)\}, \quad OBC = OCB = \tfrac{1}{2}\{\pi - (\beta + \theta - \alpha)\},$$
$$OCD = ODC = \tfrac{1}{2}\{\pi - (\alpha + \beta - \theta)\}, \quad ODA = OAD = \tfrac{1}{2}\{\pi - (\alpha + \theta - \beta)\}.$$

Hence $\quad ABC = \alpha, \quad BCD = \pi - \beta, \quad CDA = \pi - \alpha, \quad DAB = \beta.$

$$KAL = \pi - BAD = \pi - \beta = ALB + ABL = ALB + \alpha;$$

$$\therefore \quad ALB = \pi - (\alpha + \beta), \quad AB = 2r \sin \tfrac{1}{2}(\alpha + \beta + \theta),$$

$$BL = AB \sin BAL/\sin ALB = 2r \sin \tfrac{1}{2}(\alpha + \beta + \theta) \sin \beta / \sin(\alpha + \beta).$$

So $\qquad KB = 2r \sin\frac{1}{2}(\beta-\alpha+\theta)\sin\beta/\sin(\beta-\alpha),$

$$KL = (KB^2+LB^2-2KB\,.\,LB\,.\cos KBL)^{\frac{1}{2}} = C.$$

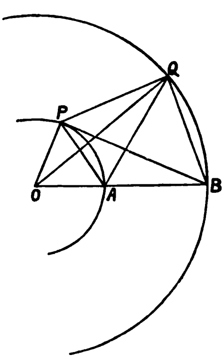
This will give θ, whence we construct the quadrilateral.

We get $\dfrac{C^2\sin^2(\beta+\alpha)\sin^2(\beta-\alpha)}{4r^2\sin^2\beta} = \sin^2\alpha\,(\cos^2\alpha+\cos^2\beta)$

$$-2\cos\beta\cos\alpha\sin\alpha\left\{\cos\beta\sin\alpha\cos(\beta+\theta)+\sin\beta\cos\alpha\sin(\beta+\theta)\right\}.$$

12663. (Rev. D. T. Griffiths, M.A.)—P and Q are any two points on the inner and outer of two concentric circles, so that PQ is of constant length. If OAB be a fixed radius drawn from the common centre O, find the locus of the intersection of AP and BQ; also of AQ and BP.

Solution by Rev. J. L. Kitchin, M.A.; Professor Sarkar; *and others.*



Let r_1, r_2 be the radii of the circles, $r_2 > r_1$, PQ $= a$, and (r_1, θ_1), (r_2, θ_2) the polar coordinates of P, Q; then

$$a^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2);$$

$\therefore\ \theta_1-\theta_2$ is constant $= \alpha$, say, and $\theta_2 = \theta_1-\alpha$.

1. The equations of AP, BQ are

$$y = \cot\tfrac{1}{2}\theta_1\,(r_1-x), \quad y = \cot\tfrac{1}{2}\theta_2\,(r_2-x);$$

$$\therefore\ \tan\tfrac{1}{2}\theta_1 = (r_1-x)/y, \quad \tan\tfrac{1}{2}\theta_2 = (r_2-x)/y;$$

$$\therefore\ \tan\tfrac{1}{2}(\theta_1-\theta_2) = \tan\tfrac{1}{2}\alpha = \frac{(r_1-r_2)\,y}{y^2+(r_1-x)(r_2-x)}$$

or $y^2+(r_1-x)(r_2-x)+(r_2-r_1)\cot\frac{1}{2}\alpha y = 0$, a circle.

2. Again, BP, AQ are $y = \dfrac{r_1\sin\theta_1}{r_2-r_1\cos\theta_1}(r_2-x)$,

$$y = \frac{r_2\sin\theta_2}{r_1-r_2\cos\theta_2}(r_1-x) = \frac{r_2\sin(\theta_1-\alpha)}{r_1-r_2\cos(\theta_1-\alpha)}(r_1-x);$$

$$\therefore\ r_1r_2\cos\theta_1 = \frac{y\left\{r_1^2(r_2-x)-r_2^2y\sin\alpha-r_2^2(r_1-x)\cos\alpha\right\}}{y(r_2-r_1)\cos\alpha-\left\{y^2+(x_1-r_1)(x-r_2)\right\}\sin\alpha}\ \ldots\ldots(A),$$

$$r_1r_2\sin\theta_1 = \frac{y\left\{r_2^2y\cos\alpha-r_2^2(r_1-x)\sin\alpha-r_1^2y\right\}}{(\ldots\ldots\ldots\ldots)}\ \ldots\ldots\ldots(B),$$

$$r_1^2r_2^2 = y^2/(\ \ )^2(A^2+B^2),$$

$$r_1^2r_2^2\left[y(r_2-r_1)\cos\alpha-\left\{y^2+(x_1-x)^2\right\}\sin\alpha\right]^2$$
$$= r_1^4\left[y^2+(r_2-x)^2+r_2^4\left\{y^2+(r_2-x)^2\right\}-2r_1^2r_2^2\{\ldots\ldots\}\right].$$

12655. (D. Biddle.)—The curved surface of a right cone is tangential to a sphere concentric with the base. Give the relative dimensions of the two bodies, (1) when their solid contents are equal, (2) when the ratio of the sphere to the cone is a maximum.

Solution by J. L. Kitchin, M.A.; Professor Chakrivarti; *and others.*

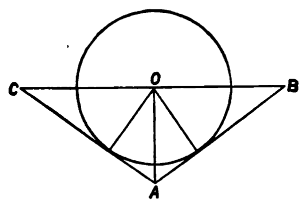
Let angle of cone $= 2\alpha$, $r =$ radius of sphere; then

$AO = r \operatorname{cosec} \alpha, \quad CO = r \sec \alpha;$

solid content of cone and sphere

$= \frac{1}{3} r^3 \pi \operatorname{cosec} \alpha \sec^2 \alpha$ and $\frac{4}{3} \pi r^3$.

(1) If these are equal, $4 \sin \alpha \cos^2 \alpha = 1$, a cubic in $\sin \alpha$, which, when solved, gives $r : AO$ and $CO : AC$.



(2) The general ratio, sphere/cone $= 4 \sin \alpha \cos^2 \alpha = U$, a maximum;

$$\therefore\ 4 \cos^3 \alpha - 8 \sin^2 \alpha \cos \alpha = 0, \quad \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) = 0.$$

$\cos \alpha = 0$ gives no result; $\cos^2 \alpha - 2 \sin^2 \alpha = 0$ gives $\sin \alpha = 1/\sqrt{3}$.

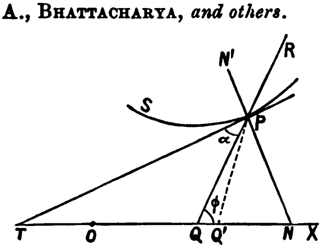
$$\text{angle of cone} = \sin^{-1} \tfrac{1}{3} (2\sqrt{2}), \quad AO = r\sqrt{3}, \quad CO = (r\sqrt{3})/\sqrt{2}.$$

3102. (Professor Hudson, M.A.)—If the path of a ray cut at a constant angle α the surfaces of equal density in a variable medium, prove that $\mu = \mu_0 \epsilon^{\phi \tan \alpha}$, where ϕ is the inclination of the path to a fixed line.

Solution by Professors Dastur, M.A., Bhattacharya, *and others.*

On the stratum SP of constant refractive power let the ray QP be incident, and let its direction after refraction be Q′PR. Ultimately, PQ is tangent to the path, and, if PT be tangent to SP, $\angle$ TPQ $= \alpha$. Let PQ make ϕ with the initial line OX.

The differential equation of refraction is $\partial\mu \cos \alpha = \mu \partial\phi \sin \alpha$ (see Heath's *Optics*, p. 334).



In the limit, $\dfrac{1}{\mu} = \tan \alpha \dfrac{d\phi}{d\mu}$; hence $\log \mu + C = \phi \tan \alpha$.

If $C = -\log \mu_0$, we have $\log (\mu/\mu_0) = \phi \tan \alpha$; $\quad \therefore\ \mu = \mu_0 \epsilon^{\phi \tan \alpha}$.

8507. (By P. C. Ward, M.A.—Given the base and vertical angle of a triangle, show that (1) the envelope of its nine-point circle is a circle concentric with the base; and hence (2), by inversion, the envelope of the circum-circle of a triangle is a circle, when the vertical angle is given in magnitude and position, and the in- or corresponding ex-circle.

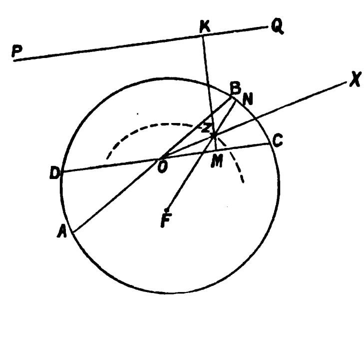
Solution by H. J. Woodall, A.R.C.S.

(1) Since the base and vertical angle are given, therefore the circumscribed circle is given (in size and position). The nine-point circle has its radius equal to half the radius of the circum-circle, and also passes through the mid-point of the base. This latter point is given, and hence the envelope of the nine-point circle is a circle concentric with the base and equal to the circum-circle.

(2) Casey proves (*Sequel*, Book vi., § iv., Prop. 12) that, "if two circles X, Y be so related that a triangle may be inscribed in X and described about Y, the inverse of X with respect to Y is the nine-point circle of the triangle formed by joining the points of contact on Y." In the question, the in-circle of ABC touches the sides at A′, B′, C′, of which A′, B′ are fixed, because the sides from A are given in position, and C′ moves on the circle. The inverse of the circum-circle ABC is the nine-point circle of A′B′C′; the envelope of this is, by (1) above, a circle concentric with A′B′. Hence the envelope of its inverse is a circle also, *i.e.*, the envelope of the circum-circle of ABC is a circle.

12397. (G. E. Crawford, M.A.)—Two chords of a circle AOB, COD intersect within the circle at O. Show how to inscribe a circle within the sector-like figure COB.

Solution by J. H. Hooker, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



Let F be centre of circle.

Bisect BOC by OX. Centre lies in OX.

Describe PQ parallel to DC at distance = radius of circle.

Describe parabola (focus F, directrix PQ) cutting OX in Z.

Produce FZ to meet circumference in N.

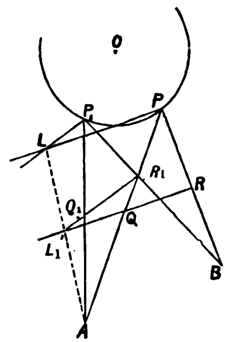
Draw KZM perpendicular to DC. Then

$$\because\ ZN = FN - FZ = KM - KZ = ZM,$$

Z is the centre of the circle.

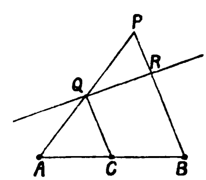
12628. (W. J. DOBBS, M.A.)—A and B are fixed points, P a movable point; Q divides AP in a fixed ratio, and QR is drawn perpendicular to PB. Prove that (1) if the locus of P is a circle, QR envelopes a fixed conic; (2) if the locus passes through B, QR passes through a fixed point; (3) if B is the centre of the locus, the conic becomes a circle; (4) if the locus is a straight line, the conic becomes a parabola.

Solution by Professor DROZ-FARNY; Rev. J. L. KITCHIN, M.A.; *and others.*



Posons $AP = n \cdot AQ$. Soient P et P_1 deux positions voisines de P sur le cercle, Q et Q_1 les positions correspondantes du point Q. Les perpendiculaires élevées en P et P_1 respectivement sur BP et BP_1 se coupent en L et les droites QR et Q_1R_1 en L_1. Les triangles PP_1L et QQ_1L_1 étant homothétiques, les points A, L_1 et L sont en ligne droite et $AL = n \cdot AL_1$. A la limite lorsque les points P et P_1 coïncident L devient le point de contact de la courbe enveloppe de PL et L_1 le point de contact de l'enveloppe de QR.

Ces deux courbes sont donc semblables, A étant le centre de similitude. Or l'enveloppe de PL d'après un théorème bien connu est une conique admettant B et son symétrique par rapport au centre O comme foyers, O comme centre et le diamètre passant par B comme grand axe. L'enveloppe (1) de QR est donc une conique dont tous les éléments sont faciles à déterminer; (2) si B appartient au cercle O, toutes les droites PL concourent au point L diamétralement opposé à B, L_1 est donc aussi un point fixe; (3) si B coïncide avec O, PL enveloppe la circonférence O même L coïncide avec P, donc L_1 avec Q; le lieu cherché sera donc une circonférence admettant avec O, A comme un des centres de similitude; (4) si P se ment sur une ligne droite, PL enveloppe une parabole il en donc aussi demême de QR.

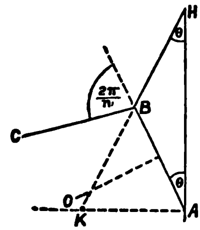


[The PROPOSER's solution is as follows:—Draw QC parallel to PB to meet AB in C; then C is a fixed point; hence the locus of Q is similar to that of P; therefore QR touches a fixed conic, with C as focus.]

12607. (Professor SANJÁNA, M.A.)—A lamina in the shape of a regular polygon of n sides is hung up against a smooth wall, to which its plane is perpendicular, by a string (attached to an angular point) equal in length to one side of the polygon. Find the position of equilibrium; and the ratios of the distances from the wall of the $n-1$ points not

supported by it. [In the simple cases $n = 3, 4, 6$, these ratios are $1:5$, $1:4:3$, $1:3:4:3:1$.]

Solution by W. J. Dobbs, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



Let AB, BC be two adjacent sides of the lamina resting against the wall at A; BH the string. Let n be the number of sides each of length $2a$, θ the angle BAH, which equals angle BHA. If HB produced meet the horizontal through A in K, for equilibrium it is necessary only that O (the centre of the lamina) should be in the same vertical line as K.

Now the distance of O from a side of the lamina is $a \cot \pi/n$; therefore the horizontal distance OA is

$$a \sin\theta + a \cot \pi/n \,.\, \cos\theta.$$

Also $\text{KA} = 4a \sin\theta$; therefore for equilibrium we must have

$$4 \sin\theta = \sin\theta + \cot \pi/n \,.\, \cos\theta,$$

which gives $\cot\theta = 3 \tan \pi/n$, determining the position of equilibrium.

The distances of B, C, ... from AH are respectively

$$2a \sin\theta,\ 2a\left\{\sin\theta + \sin(\theta + 2\pi/n)\right\},\ 2a\left\{\sin\theta + \sin(\theta + 2\pi/n) + \sin(\theta + 4\pi/n)\right\},$$
$$\ldots\ 2a\left\{\sin\theta + \sin(\theta + 2\pi/n) + \ldots + \sin\left[\theta + (n-2)/n \,.\, 2\pi\right]\right\}.$$

These distances are proportional to

$$\sin \pi/n \,.\, \sin\theta,\quad \sin 2\pi/n \,.\, \sin(\theta + \pi/n),\quad \sin 3\pi/n \,.\, \sin(\theta + 2\pi/n),$$
$$\ldots\ \sin(n-1)/n \,.\, \pi \sin\left[\theta + (n-2)/n \,.\, \pi\right];$$

i.e., proportional to

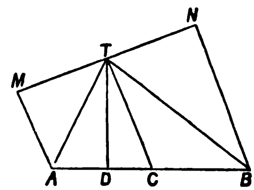
$$\sin \pi/n,\quad \sin 2\pi/n \,.\, (\cos \pi/n + \cot\theta \sin \pi/n),$$
$$\sin 3\pi/n \,.\, (\cos 2\pi/n + \cot\theta \sin 2\pi/n),$$
$$\ldots\ \sin(n-1)/n \,.\, \pi \left[\cos(n-2)/n \,.\, \pi + \cot\theta \sin(n-2)/n \,.\, \pi;\right]$$

i.e.,

$$\sin \pi/n,\ \sin 2\pi/n\,(\cos \pi/n + 3 \tan \pi/n \sin \pi/n),$$
$$\sin 3\pi/n\,(\cos 2\pi/n + 3 \tan \pi/n \,.\, \sin 2\pi/n),$$
$$\ldots\ \sin(n-1)/n \,.\, \pi \left[\cos(n-2)/n\, \pi + 3 \tan \pi/n \sin(n-2)/n \,.\, \pi.\right]$$

12620. (Professor Krishnachandra De, M.A.)—A variable straight line moves in such a manner that the difference of the squares on its distances from two given points A, B is constant $(= K^2)$. Prove, by elementary geometry, that the envelope of the variable line is a parabola whose semi-parameter is equal to K^2/AB.

Solution by REV. J. L. KITCHIN, M.A.; H. W. CURJEL, M.A.; *and others.*



Let C be the middle point AB and MTN the variable straight line. Draw AM, CT, BN perpendicular to MTN and TD perpendicular to AB. Then MT = TN, and

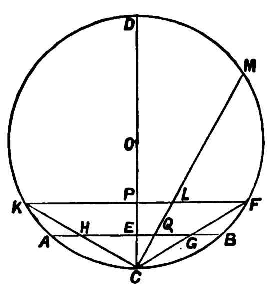
$K^2 = BN^2 - AM^2 = BD^2 - AD^2 = 2AB.DC$;

therefore D is a fixed point; therefore the envelope of MN is a parabola with focus C, vertex D, and semi-latus-rectum

$$= 2DC = K^2/AB.$$

12393. (J. MACLEOD, M.A.)—Taking the property of Euc. III. 35 as the definition of a circle, show that the ordinary definition may be established therefrom as a theorem.

Solution by the PROPOSER; Professor KRISHNACHANDRA; *and others.*



Bisect the chord AB in E, through which the perpendicular CD is drawn and bisected in O. We have $CE.ED + EO^2 = CO^2$ for all positions of E. Take the chord CGF cutting AB in G, and having cut off EH = EG, and CH being joined, is produced to meet the circumference in K.

By definition,

$CH.HK = AH.HB = AG.GB = CG.GF$,

and, as CH = CG, we evidently have CK = CF. If KF cut CD in P, since $\angle KCP$ is evidently $= \angle FCP$, we have $\angle CPK$ and $\angle CPF$ right angles, and KP = PF.

Let us now draw any chord CLM cutting KF in L, and we get

$$KL.LF + PL^2 = CL.LM + PL^2.$$

But $KL.LF + PL^2 + OP^2 = CL.LM + OL^2 = KP^2 + OP^2 = OC^2$.

It is true, therefore, for any point L in a chord drawn through O, that

$$CL.LM + OL^2 = OC^2.$$

Let L be the foot of the perpendicular from O on CM, and we get

$$CL.LM + OL^2 = CQ.QM + OQ^2 = CQ.QM + OL^2 + LQ^2$$

or $CL.LM = CQ.QM + LQ^2$; $\therefore$ CL = LM; $\therefore$ OM = OC.

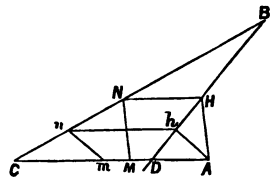
12637. (I. ARNOLD.)—Two given straight lines AC, BC meet one another in C; draw a straight line MN cutting AC in M and BC in N, so that AM may be equal to half BN, and MN a minimum.

Solution by W. J. Dobbs, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Take M, N in any position, such that AM = $\frac{1}{2}$BN; and complete the parallelogram AMNH.

Then NH = $\frac{1}{2}$NB; therefore the locus of H is a straight line.

Hence the following construction:—Along CA make CD = $\frac{1}{2}$CB. Draw perpendicular Ah upon BD. Draw hn parallel to AC to meet BC in n, and draw nm parallel to Ah to meet AC in m. Then mn is the line required.



3359. (Rev. F. D. Thomson, M.A.)—At the election of a school board of nine members, there are eleven candidates. Show that each elector may dispose of his nine votes in 92378 different ways.

Solution by Professors Sanjána, Lampe, *and others.*

The nine votes may all be given to one candidate; or eight of them to one and the ninth to another; or seven of them to one and the remaining two to one or two of the remaining candidates; and so on. Thus the number required is $_{11}H_9 = \dfrac{19!}{10!\,9!} = \dfrac{11.12.13.14.15.16.17.18.19}{1.2.3.4.5.6.7.8.9} = 92378.$

12578. (Professor Zerr.)—A. runs round the circumference of a circular field with velocity m feet; B. starts from the centre with velocity n ($>m$) feet to catch A. The straight line joining their positions always passes through the centre. Find the equation to the curve described by B., the distance he runs, and the time occupied.

Solution by Professors Sanjána, Bhattacharya, *and others.*

Let P, P′ be two successive positions of A, corresponding to Q, Q′ of B, so that BQP, BQP′ are straight lines.

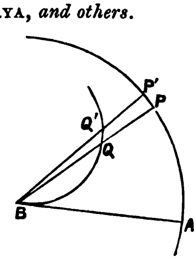
Let ABP = θ, PBP′ = $\delta\theta$, AB = a, BQ = r,

Then PP′ = $a\delta\theta$; QQ′ = $\left\{(\delta r)^2+(r\delta\theta)^2\right\}^{\frac{1}{2}}$;

but PP′ = QQ′ = $m : n$. Hence

$$n^2a^2(\delta\theta)^2 = m^2(\delta r)^2 + m^2r^2(\delta\theta)^2.$$

In the limit we obtain $\left(\dfrac{d\theta}{dr}\right)^2 = \dfrac{m^2}{a^2n^2-m^2r^2}$;



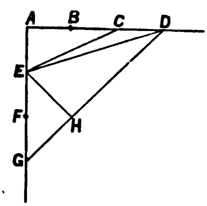
$$\therefore \quad \theta = \int \frac{mdr}{(a^2n^2 - m^2r^2)^{\frac{1}{2}}} = \sin^{-1}\frac{mr}{an}.$$

Hence $r = an/m \sin\theta$ is the equation of the locus of Q, which is thus seen to be a circle touching BA at B.

It will be found that the length of path when B catches A is $an/m \sin^{-1} m/n$, and the time required is $a/m \sin^{-1} m/n$.

12613. (Professor MOREL.)—Sur l'un des côtés d'un angle droit, on porte successivement, à partir du sommet A, les longueurs égales $AB = BC = CD = a$, et sur l'autre côté, les longueurs égales $AE = EF = FG = a$; on tire les droites CE, ED, DG. Prouver que l'angle ACE est égal à l'angle EDG.

Solution by Rev. S. J. ROWTON, M.A.; Prof. AIYAR; *and others.*



Draw EH perpendicular to GD. Then

$$\sin ACE = \frac{AE}{EC} = \frac{a}{a\sqrt{5}} = \frac{1}{\sqrt{5}},$$

and $$\sin EDG = \frac{EH}{ED} = \frac{a\sqrt{2}}{a\sqrt{10}} = \frac{1}{\sqrt{5}};$$

therefore $\angle ACE = \angle EDG$, since each is acute, being evidently $< \frac{1}{2}$ right angle.

12412. (Professor LECTA MILLER.)—Bought sugar at $6\frac{1}{2}$ cents a pound; waste by transportation and retailing was 5 per cent.; interest on first cost to time of sale was 2 per cent. Find how much must be asked per pound to gain 25 per cent.

Solution by R. CHARTRES, Professor SHIELDS, *and others.*

$\frac{19}{20}$ lb. must be sold for $\frac{13}{2} \times \frac{51}{50} \times \frac{5}{4}$ cents, or 1 lb. for

$$\frac{13 \times 51}{4 \times 19} = 8\tfrac{55}{76} \text{ cents.}$$

12394. (R. TUCKER, M.A.)—Evaluate

$$\begin{vmatrix} \cos A - \cos 2A\cos B-C, & 2\cos^2 A\cos B, & 2\cos^2 A\cos C \\ 2\cos^2 B\cos A, & \cos B - \cos 2B\cos C-A, & 2\cos^2 B\cos C \\ 2\cos^2 C\cos A, & 2\cos^2 C\cos B, & \cos C - \cos 2C\cos A-B \end{vmatrix}.$$

Solution by Rev. T. Roach, M.A.; Professor Sanjana; *and others.*

The determinant

$$= 2\cos^3 A \cos 2B\, 2\sin C \sin A \cos 2C\, 2\sin A \sin B$$
$$+ 2\cos^3 B \cos 2C\, 2\sin A \sin B \cos 2A\, 2\sin B \sin C$$
$$+ 2\cos^3 C \cos 2A\, 2\sin B \sin C \cos 2B\, 2\sin C \sin A$$
$$-8\cos 2A \sin B \sin C \cos 2B \sin C \sin A \cos 2C \sin A \sin B$$
$$= 2\sin A \sin B \sin C \sin 2A \sin 2B \sin 2C.$$

12502. (Professor Neuberg.)—Intégrer l'équation

$$y(z-b)-(y-a)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=0.$$

Solution by H. W. Curjel, M.A.; Professor Bhattacharya; *and others.*

The equation may be written

$$x\frac{\partial \log(z-b)}{\partial x}+y\frac{\partial \log(z-b)}{\partial y}=\frac{y}{y-a};$$

therefore the general integral is

$$\log(z-b)=\phi(x/y)+\log(y-a); \quad \therefore\ z=(y-a)f(x/y)+b,$$

where $f(x/y)$ is an arbitrary function of x/y.

12484. (Rev. T. Roach, M.A.)—Prove that

$$\frac{1}{1.3.5}+\frac{1}{13.15.17}+\ldots=\frac{\log 3}{32}+\frac{\pi}{96},\quad \frac{1}{7.9.11}+\frac{1}{19.21.23}+\ldots=\frac{\log 3}{32}-\frac{\pi}{96},$$

$$\frac{1}{1.3.5}-\frac{1}{7.9.11}+\frac{1}{13.15.17}-\ldots=\frac{\pi}{48}.$$

Solution by J. J. Barniville, B.A.; H. W. Curjel, M.A.; *and others.*

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots=\tfrac{1}{2}\log\frac{1+x}{1-x},\quad 3\left(\frac{x^3}{3}+\frac{x^9}{9}+\frac{x^{15}}{15}+\ldots\right)=\tfrac{1}{2}\log\frac{1+x^3}{1-x^3};$$

therefore $$x-\frac{2x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{2x^9}{9}+\ldots=\tfrac{1}{2}\log\frac{1+x+x^2}{1-x+x^2}.$$

Making $x=1$, we get

$$1-\tfrac{2}{3}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{2}{9}+\tfrac{1}{11}+\ldots=\tfrac{1}{2}\log 3,$$

i.e., $$\frac{1}{1\,.\,3\,.\,5}+\frac{1}{7\,.\,9\,.\,11}+\frac{1}{13\,.\,15\,.\,17}+\ldots=\tfrac{1}{16}\log 3.$$

Also $$\frac{1}{1\,.\,3\,.\,5}-\frac{1}{7\,.\,9\,.\,11}+\frac{1}{13\,.\,15\,.\,17}-\ldots=\tfrac{1}{8}\left\{1-\tfrac{2}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{2}{9}-\ldots\right\}$$

$$=\tfrac{1}{8}\left\{1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\right\}=\pi/48\,;$$

adding and subtracting, we obtain the required results. Similarly,

$$\tfrac{1}{2}\log 5=1+\tfrac{1}{3}-\tfrac{4}{5}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{4}{15}+\ldots.$$

12539. (Professor HUDSON, M.A.)—An irrigation canal discharges 7 tons of water per acre per day on 3,000,000 acres; find (1) how many tons it discharges per second, and (2) how many lbs. per square yard per annum.

Solution by T. SAVAGE, Professor AIYAR, *and others.*

(1.) $\dfrac{7\times 3000000}{24\times 60\times 60}=243\tfrac{1}{18}$ tons per second;

(2.) $\dfrac{7\times 2240\times 365}{4840}=1182\tfrac{58}{121}$ lbs. per sq. yd. per annum.

11673. (H. J. WOODALL, A.R.C.S.)—Place 4 sovereigns and 4 shillings in close alternate order in a line. Required in four moves, each of two contiguous pieces (without altering the relative positions of the two), to form a *continuous* line of 4 sovereigns followed by 4 shillings.

Solution by Rev. S. J. ROWTON, M.A., Mus.D.

Let $aAbBcCdD$ be the original order, the small letters representing the shillings and the capitals the sovereigns. (Apparently we are not required to begin with a sovereign on the left.) Then we get

$$\text{either}\ \left\{\begin{array}{l} \mathrm{C}d\,a\,\mathrm{A}b\mathrm{B}c\,.\,.\,\mathrm{D}, \\ \mathrm{C}d\,a\,\mathrm{A}\,.\,.\,cb\mathrm{BD}, \\ \mathrm{C}\,.\,.\,\mathrm{A}dacb\mathrm{BD}, \\ \mathrm{CBDA}dacb, \end{array}\right. \qquad \text{or}\ \left\{\begin{array}{l} a\,.\,.\,\mathrm{B}\,c\,\mathrm{C}d\mathrm{DA}b, \\ ac\mathrm{CB}\,.\,.\,d\mathrm{DA}b, \\ ac\mathrm{CBDA}d\,.\,.\,b, \\ \quad\mathrm{CBDA}da\;c\;b. \end{array}\right.$$

The dots denote spaces left temporarily vacant.

The solution offered by Mr. DAVIS (Vol. LIX., p. 45) obviously fails to comply with the conditions.

[The puzzle is more difficult if the question is put in the converse way.]

12625. (H. FORTEY.)—Let

$$f(x) = x^{2(p-2)n} + x^{2(p-3)n} + \&c. + x^{2n} + 1 + (x^{p-2} + x^{p-3} + \&c. + x + 1)^{2n},$$

and
$$\phi(x) = x^{p-1} + x^{p-2} + \&c. + x + 1;$$

then, if p be a prime number greater than 2, and n a positive integer not a multiple of p, (1) $f(x)$ is divisible by $\phi(x)$; (2) if n be of the form $pr - \frac{1}{2}(p-1)$, $f(x)$ is divisible by $[\phi(x)]^2$.

Solution by H. W. CURJEL, M.A.; Professor BHATTACHARYA; *and others.*

Let a be a root of $\phi(x) = 0$; then $a^p = 1$, and a^{2n} is also a root; hence

$$f(a) = -a^{2n(p-1)} + (-a^{p-1})^{2n} = 0;$$

but $\phi(x) = 0$ has no double root; therefore $f(x)$ is divisible by $\phi(x)$.

If $n = pr - \frac{1}{2}(p-1)$, then $2n \equiv 1 \mod p$, and

$$\begin{aligned} f'(a) &= 2n(p-2)a^{2n(p-2)-1} + 2n(p-3)a^{2n(p-3)-1} + \ldots + 2na^{2n-1} \\ &\quad + 2n(a^{p-2} + a^{p-3} + \ldots + 1)^{2n-1}\left\{(p-2)a^{p-3} + (p-3)a^{p-4} + \ldots + 1\right\} \\ &= 2n(p-2)a^{p-3} + 2n(p-3)a^{p-4} + \ldots + 2n \\ &\quad - 2n(a^{p-1})^p\left\{(p-2)a^{p-3} + (p-3)a^{p-4} + \ldots + 1\right\} = 0; \end{aligned}$$

therefore $f(x)$ is divisible by $\left\{\phi(x)\right\}^2$, when $n = pr - \frac{1}{2}(p-1)$.

12363. (S TEBAY, B.A.)—If the equation $x^3 + px + q = 0$ be written down at random, prove that the probability that the roots are real is $\frac{1}{4}$.

Solution by Professor AIYAR; *the* PROPOSER; *and others.*

If the roots be real and denoted by a, b, c, we have

$$a + b + c = 0, \quad \text{and} \quad (a+b+c)^2 = a^2 + b^2 + c^2 + 2(bc + ca + ab) = 0;$$

but
$$bc + ca + ab = p; \quad \text{therefore } a^2 + b^2 + c^2 + 2p = 0;$$

and, since $a^2 + b^2 + c^2$ is absolutely positive, therefore p is negative.

If p be positive, the equation contains unreal roots; for, if the roots be all real, p is negative. But the equation may contain unreal roots when p is negative.

Let
$$a = \alpha + \beta i, \quad b = \alpha - \beta i, \quad c = -2\alpha, \quad \text{where } i = \sqrt{(-1)}.$$

Then
$$bc + ca + ab = -3\alpha^2 + \beta^2;$$

which may be either positive or negative.

Now the probability that p is negative is $\frac{1}{2}$; when p is negative the probability that the roots are real is $\frac{1}{2}$. Hence the joint probability is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

10183. (Professor RAMASWAMI AIYAR.)—Prove that the centre of gravity of equal particles placed at the vertices of similar polygons of any species whatever, similarly described on the sides of a given polygon, is a fixed point.

Solution by H. J. WOODALL, A.R.C S.; Prof. BHATTACHARYA; *and others.*

Let ABCD ... N be the given polygon, and let

$Aa_1b_1c_1 \dots k_1B$, $Ba_2b_2c_2 \dots k_2C$, $Ca_3b_3c_3 \dots k_3D$, ... $Na_nb_nc^n \dots k_nA$,

be the similar polygons described on the sides of the given polygon. Now join the points thus $a_1a_2a_3 \dots a_n$, $b_1b_2b_3 \dots b_n$, ... $k_1k_2k_3 \dots k_n$, and we get new polygons which are all similar to the given polygon ABC...N; they will also be similarly placed; the one common point being the C.G. of the vertices, and this C.G. is the fixed point spoken of.

12516 & 12609. (Professor SHIELDS.)—Each of five men, A, B, C, D, E, owned a different sum of money, each kept one-half of his money, and each in succession invested the other half in different-priced lottery tickets, on this condition, that he who draws and loses shall pay to each of the others as much as they already have. First A draws and loses, paying to each of the others as much as they already have, then B draws and loses, paying to each of the others as much as they already have, then C, then D, and last also E; all draw and lose in turn, and yet, after paying each of the others as much as they already had at the end of each drawing, they have all the same sum of money, viz., \$16 each. Find (1) the cost of each man's ticket, and (2) how much each party gained or lost.

Solution by D. BIDDLE, T. SAVAGE, Rev. J. L. KITCHIN, M.A., *and others.*

Let α, β, γ, δ, ϵ be the cost of the respective tickets. Then, at the termination of the several distributions, A, B, C, D, E possess the following sums:—

I. $\alpha-\beta-\gamma-\delta-\epsilon,\ 2\beta,\ 2\gamma,\ 2\delta,\ 2\epsilon;$

II. $2(\alpha-\beta-\gamma-\delta-\epsilon),\ 3\beta-\alpha-\gamma-\delta-\epsilon,\ 4\gamma,\ 4\delta,\ 4\epsilon;$

III. $4(\alpha-\beta-\gamma-\delta-\epsilon),\ 2(3\beta-\alpha-\gamma-\delta-\epsilon),\ 7\gamma-\alpha-\beta-\delta-\epsilon,\ 8\delta,\ 8\epsilon;$

IV. $8(\alpha-\beta-\gamma-\delta-\epsilon),\ 4(3\beta-\alpha-\gamma-\delta-\epsilon),\ 2(7\gamma-\alpha-\beta-\delta-\epsilon),$
$15\delta-\alpha-\beta-\gamma-\epsilon,\ 16\epsilon;$

V. $16(\alpha-\beta-\gamma-\delta-\epsilon),\ 8(3\beta-\alpha-\gamma-\delta-\epsilon),\ 4(7\gamma-\alpha-\beta-\delta-\epsilon),$
$2(15\delta-\alpha-\beta-\gamma-\epsilon),\ 31\epsilon-\alpha-\beta-\gamma-\delta.$

And each sum, in the last, = \$16, whence

$$\alpha = 40\tfrac{1}{2},\quad \beta = 20\tfrac{1}{2},\quad \gamma = 10\tfrac{1}{2},\quad \delta = 5\tfrac{1}{2},\quad \epsilon = 3.$$

Moreover, A loses \$65, B loses \$25, C loses 5\$, D gains \$5, and E gains \$10.

Generally, n being the number of persons, and 2^{n-1} being the amount

each has at the end, the several sums possessed at the beginning are $2^{n-1}.n+1$, $2^{n-2}.n+1$, $2^{n-3}.n+1$, ... $2^0.n+1$, the cost of the ticket being half such sum in each instance.

12533. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{1.3.5}-\frac{1}{4.6.8}+\frac{1}{7.9.11}-\ldots=\frac{1}{16},$$

$$\frac{1}{1.3.5}+\frac{1}{7.9.11}+\frac{1}{13.15.17}+\ldots=\frac{1}{16}\log 3,$$

$$\frac{1}{1.9.17}-\frac{1}{13.21.29}+\frac{1}{25.33.41}-\ldots=\frac{\log(1+\surd 2)}{128\surd 2}+\frac{1}{640}.$$

Solution by H. W. Curjel, M.A.; Rev. T. Roach, M.A.; *and others.*

As in Solution to Quest. 12484, the second series $=\frac{1}{16}\log 3$.

Similarly, $$8\left\{\frac{1}{4.6.8}+\frac{1}{10.12.14}+\ldots\right\}$$

$$=\Sigma_1^\infty 1/(3n-1)\,3n\,(3n+1)=\tfrac{1}{2}\Sigma_1^\infty\left\{\frac{x^{3n-1}}{3n-1}+\frac{x^{3n}}{3n}+\frac{x^{3n+1}}{3n+1}-\frac{x^{3n}}{n}\right\}$$

where $x=1$

$$=-\tfrac{1}{2}x-\tfrac{1}{2}\log(1-x)+\tfrac{1}{2}\log(1-x^3)=-\tfrac{1}{2}x-\tfrac{1}{2}\log(1+x+x^2)$$

$$=-\tfrac{1}{2}+\tfrac{1}{2}\log 3;\qquad \therefore\ \text{first series}=\tfrac{1}{16}\log 3-(\tfrac{1}{16}\log 3-\tfrac{1}{16})=\tfrac{1}{16}.$$

The third series

$$=\Sigma_1^\infty\left\{\frac{1}{(24n-23)(24n-15)(24n-7)}-\frac{1}{(24n-11)(24n-3)(24n+5)}\right\}$$

$$=\frac{1}{128}\Sigma_1^\infty\left\{\frac{1}{24n-23}+\frac{1}{24n-15}+\frac{1}{24n-7}-\frac{1}{8n-5}-\frac{1}{24n-11}-\frac{1}{24n-3}-\frac{1}{24n+1}+\frac{1}{8n-1}\right\}$$

$$=\frac{1}{128}\left\{\frac{1}{5}+\frac{1}{1}-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\frac{1}{17}-\frac{1}{21}+\ldots-\frac{1}{3}+\frac{1}{7}-\frac{1}{11}+\ldots\right\}$$

$$=\frac{1}{640}+\frac{1}{128\times 4}\left\{\left(\alpha+\frac{1}{\alpha}\right)\log\frac{1+\alpha}{1-\alpha}+\left(\beta+\frac{1}{\beta}\right)\log\frac{1+\beta}{1-\beta}\right\}$$

[where $\alpha^2=\surd(-1)$, $\beta^2=-\surd(-1)$]

$$=\frac{1}{640}+\frac{1}{128\times 4}\left\{\frac{1+i}{\alpha}\log\frac{1+i+2\alpha}{1-i}+\frac{1-i}{\beta}\log\frac{1-i+2\beta}{1+i}\right\}$$

$$=\frac{1}{640}+\frac{1}{128\times 4}\left\{\surd 2\log\frac{(1+i)(1+\surd 2)}{1-i}+\surd 2\log\frac{(1-i)(1+\surd 2)}{1+i}\right\}$$

$$=\frac{1}{640}+\frac{1}{128\times 4}\left\{\surd 2\log(1+\surd 2)^2\right\}=\frac{1}{640}+\frac{\log(1+\surd 2)}{128\surd 2}.$$

12686. (Rev. T. C. SIMMONS, M.A.)—In the expansion of $(a+1)^{(na+1)}$, prove that the sum of the first n terms exceeds the sum of the last na terms.

Solution by D. BIDDLE.

Lying between the two specified sets of terms, the $(n+1)$th or neutral term $= \dfrac{(na+n)!\,a^{na}}{n!\,(na)!} = \text{M}$. The nth and the $(n+2)$th terms are respectively $na/(na+1)\,.\,\text{M}$ and $na/(na+a)\,.\,\text{M}$, the ratio of the former to the latter being $na+a : na+1$. Consequently, when $a>1$, the nth term exceeds the $(n+2)$th term by $(a-1)/(na+1)$ of the latter; but M is greater than either, and, in fact, is the maximum term of the expansion.

Let the numerator of M be represented by $k = (na+n)!\,a^{na}$, and let x represent the positional number backwards or forwards from M of any particular term of the expansion; then such term, if before M,

$$= k\,.\,a^x/\{(n-x)!\,(na+x)!\},$$

and, if after M, $\quad = k/\{(n+x)!\,(na-x)!\,a^x\}$.

The theorem, therefore, resolves itself into

$$\Sigma_1^n \frac{a^x}{(n-x)!\,(na+x)!} > \Sigma_1^{na} \frac{1}{(n+x)!\,(na-x)!\,a^x} \quad \ldots\ldots\ldots\ldots (1).$$

But the best and simplest way is to regard the several terms as weights, placed at equal distances on a straight line $na+n$ in length. Then, by taking their moments about one extremity, we find that the centre of mass is at a distance n from the first term, and coincides with the $(n+1)$th term. Thus

$$\frac{0\,.\,a^{na+n}+1\,.\,(na+n)\,a^{na+n-1}+(na+n)(na+n-1)\,a^{na+n-2}+\ldots}{a^{na+n}+(na+n)\,a^{na+n-1}+\frac{1}{2}(na+n)(na+n-1)\,a^{na+n-2}+\ldots}$$
$$= \frac{(na+n)\,(a+1)^{na+n-1}}{(a+1)^{na+n}} = n \quad \ldots (2).$$

Next, we are able to take the moments of the terms on each side of the $(n+1)$th, about that term, and find the sum of each set of moments to be identical. We can therefore declare that, multiplying the original terms backwards and forwards from the $(n+1)$th by 1, 2, 3, 4, &c., in succession, and then adding the products of each set, we get two sums which are equal, although the na terms beyond are more numerous than the n terms preceding the $(n+1)$th.

The following scheme will graphically represent what we have in the expansion of $(a+1)^{n(a+1)}$ when $a=2$ and $n=3$.

$$\left.\begin{array}{r} \text{A}+\text{B}+\text{C} \\ +\text{A}+\text{B} \\ +\text{A} \end{array}\right\} \begin{array}{c} +\text{M}+ \\ = \\ \end{array} \left\{\begin{array}{r} \text{D}+\text{E}+\text{F}+\text{G}+\text{H}+\text{I} = (a+1)^{n(a+1)} \\ +\text{E}+\text{F}+\text{G}+\text{H}+\text{I} \\ +\text{F}+\text{G}+\text{H}+\text{I} \\ +\text{G}+\text{H}+\text{I} \\ +\text{H}+\text{I} \\ +\text{I} \end{array}\right. \quad \ldots\ldots\ldots (3).$$

We have already seen that $\text{C}>\text{D}$; and, if x represent the distance from M of any term to left or right, the proportion to it of the one more remote from M, but next in order, is given by $a(n-x)/(na+x+1)$ for those on

the left, and by $(na-x)/\{a(n+x+1)\}$ for those on the right. We are thus able to give for both sides a numerator and denominator of which the product of x factors gives the particular term.

$$\left.\begin{array}{ll} \text{Left} & \dfrac{M.n \quad . \quad (n-1) \quad . \quad (n-2) \quad . \quad (n-3) \quad . \quad (n-4) \quad \ldots}{\left(n+\frac{1}{a}\right).\left(n+\frac{2}{a}\right).\left(n+\frac{3}{a}\right).\left(n+\frac{4}{a}\right).\left(n+\frac{5}{a}\right)\ldots} \\ & \qquad (1) \qquad\quad (2) \qquad\quad (3) \qquad\quad (4) \qquad\quad (5) \\ \text{Right} & \dfrac{M.n.\left(n-\frac{1}{a}\right).\left(n-\frac{2}{a}\right).\left(n-\frac{3}{a}\right).\left(n-\frac{4}{a}\right)\ldots}{(n+1). \quad (n+2) \quad . \quad (n+3) \quad . \quad (n+4) \quad . \quad (n+5) \quad \ldots} \end{array}\right\} \ldots (4).$$

We can also similarly represent by continuous factors the ratio of the corresponding terms (left : right).

$$\dfrac{\overset{(1)}{\{n+1\}} \; . \; \overset{(2)}{\{(n-1)(n+2)\}} \; . \; \overset{(3)}{\{(n-2)(n+3)\}} \; \ldots}{\left\{n+\frac{1}{a}\right\}.\left\{\left(n-\frac{1}{a}\right)\left(n+\frac{2}{a}\right)\right\}.\left\{\left(n-\frac{2}{a}\right)\left(n+\frac{3}{a}\right)\right\}\ldots} \quad \ldots\ldots\ldots\ldots\ldots \; (5).$$

At the commencement the ratio exceeds unity, but it constantly declines, the separate factor (numerator and denominator) falling below

unity when $$x > \tfrac{1}{2}\left\{1+\left(\frac{a+1+4na}{a+1}\right)^{\frac{1}{2}}\right\},$$

and the ratio itself quickly following, when $n > 2 + (1/a)$.

Such being the case, let us revert to the scheme (3). If, in the first row, the terms to the right of M are *not* less than those to the left, those in the second row are greater because the removal of C takes more away from the left than the removal of D takes from the right. And each succeeding removal makes the remainder on the left bear a lower proportion to that on the right. For it has been shown that C > D, and C/D > B/E > A/F > &c.; so that in each succeeding row the ratio between the respective first terms, counting from the mid-space of (3), becomes increasingly disadvantageous to the left side; so likewise does the ratio between the respective second terms (only more so), and so on, the terms being also of declining value on both sides. When, as shown above, the separate terms on the *left* become less than the corresponding terms on the *right*, there can be no doubt R > L, although R − L will then grow less. The difference so expressed never can become a minus quantity, and R/L will increase still. Moreover, when the terms on the left are exhausted, there are still $(na-n)$ rows on the right untouched, though dwindling in size and importance. And the superiority of R over L in the lower portion of the scheme is made still clearer when we reverse the process by adding terms from the foot upwards until the maximum R − L is reached. But the sum of the rows on the right of M is equal to the sum of the rows on the left. Hence, &c., by a *reductio ad absurdum*.

As to the sum S of the "moments" about the "centre of mass," if, having multiplied the terms by 1, 2, 3, 4, &c., as above, we take them

in order from the beginning of the expansion to the nth term, we have

$$S = n \,.\, a^{na+n} + (n-1)(na+n)a^{na+n-1} + (n-2)(na+n)\left\{\tfrac{1}{2}(na+n-1)\right\}a^{na+n-2} + \ldots,$$

or, $S = n(na+n-1)\left\{\tfrac{1}{2}(na+n-2)\right\}\left\{\tfrac{1}{3}(na+n-3)\right\}\ldots a^{na}$

$= na$ times the $(n+1)$th term in the expansion of a^{na+n-1} ...... (6),

and, as this term is $1/(a+1)$ of the $(n+1)$th term in the expansion of $(a+1)^{na+n}$, $\qquad S = M \,.\, na/(a+1)$ .... (7).

This is important, as enabling us to sum the two following series:—

$$\Sigma_1^n \frac{x \,.\, a^x}{(n-x)!\,(na+x)!} = \Sigma_1^{na} \frac{x}{(n+x)!\,(na-x)!\,a^x} = \frac{na}{a+1} \cdot \frac{1}{(na)!\,n!} \quad \ldots (8).$$

If we now fill up, in (3), the n rows on the left, and the na rows on the right of M, the added portions will be multiples, by $(na+n)$ and $(na+n)a$, respectively, of $(n-1)$ terms from the beginning and $(na-1)$ terms from the end of the expansion of $(a+1)^{na+n-1}$.

Let P, Q be the respective sums of original terms to left and right of M in the expansion of $(a+1)^{na+n}$, and let P′, Q′ be the $(n-1)$ terms, and the $(na-1)$ terms above referred to in the expansion of a^{na+n-1}. P′ and Q′ are separated by two terms which are always of identical value, $= M/(a+1) = M'$. We therefore have

$$P + Q = (a+1)(P' + M' + Q') = \left\{(a+1)P' + aM'\right\} + \left\{(a+1)Q' + M'\right\} \ldots (9).$$

But it is easy to see from (2) that, if, starting from the next to M, we multiply the terms in Q by $n+1$, $n+2$, &c., we get multiples, by $(na+n)$, of corresponding terms in the expansion of $(a+1)^{na+n-1}$. This process, however, gives us $nQ + S = (na+n)(Q' + M')$, whence $Q = (a+1)\,Q' + M'$,

and, by (9), $\qquad P = (a+1)\,P' + aM'$ .... (10).

$$\therefore \quad P - Q = (a+1)(P' - Q') + (a-1)\,M',$$

and $\qquad (P-Q)/(a+1) = P' - Q' + M'\,(a-1)/(a+1)$ .... (11).

But, taking the terms to right and left of the $(n+1)$th in the lower expansion, $\quad (P-Q)/(a+1) = \left\{(P' + M') - Q'\right\} - 2M'/(a+1)$ .... (12).

And, if we examine the relation between $(P+M+Q)_{n(a+1)}$ and $(P+M+Q)_{(n-x)(a+1)}$, we shall find that the ratio between the respective terms represented by M is much less than that between the total expansions, and that the ratio of $(P-Q)_{n(a+1)}$ to $(P-Q)_{(n-x)(a+1)}$ lies between the other two, and much nearer the M-ratio than the expansion-ratio. Consequently, as n increases, although $P-Q$ becomes actually greater, it becomes proportionately less as compared with the total expansion: it is proportionately greatest when $n = 1$.

The ratio of $M_{n(a+1)}$ to $M_{(n-x)(a+1)}$ is

$$\frac{\left\{n\,(a+1)\right\}!\,a^{na}}{(na)!\,n!} : \frac{\left\{(n-x)(a+1)\right\}\,a^{(n-x)a}}{\left\{(n-x)\,a\right\}!\,(n-x)!}$$

$$= \frac{\left\{n\,(a+1)\right\}!\,\left\{(n-x)\,a\right\}!\,(n-x)!\,a^{xa}}{\left\{(n-x)(a+1)\right\}!\,(na)!\,n!},$$

whereas the ratio of $(a+1)^{n(a+1)}$ to $(a+1)^{(n-x)(a+1)}$ is $(a+1)^{x(a+1)}$.

[The fact that the difference becomes proportionately less as n increases was first announced by the PROPOSER to the London Mathematical Society, on March 14, 1895. See *Educational Times* for April, 1895, p. 200.]

12650. (C. L. DODGSON, M.A.)—To discover the rule by which the following puzzle is worked. It is best exhibited as a dialogue.

A. Think of a number less than 90.—B. I have done so.

A. Tack on to it any digit you like, from 0 to 9. Which shall it be?—B. I have tacked on a 7.

A. Now divide by 3. What is the remainder?—B. It is 2.

A. Tack on to the quotient any digit you like.—B. I have tacked on 4.

A. Divide by 3. What is the remainder?—B. It is 1.

A. And what is the third figure from the end?—B. It is 8.

A. (Instantly rejoins) Then the number you thought of was 76.

Solution by D. BIDDLE, Professor RADHAKINSHUAN, *and others.*

Let $10x+y$ = the required number, whilst a, b, c, d, e are the given figures in the order of their declaration. Then we have

$$\tfrac{1}{3}(100x+10y+a-b) = 33x+3y+\tfrac{1}{3}(x+y+a-b) = Q_1,$$

and $\tfrac{1}{9}\left\{1000x+100y+10a-10b+3(c-d)\right\}$

$$= 111x+11y+a-b+\tfrac{1}{9}\left\{x+y+a-b+3(c-d)\right\} = Q_2.$$

We therefore know that $x+y+\left\{a-b+3(c-d)\right\}$ is divisible by 9. Moreover, since 89 has the maximum $(x+y)$, we know that $(x+y)$ cannot exceed 17. Another known fact is that the third figure from the end of Q_2, represented by e, must either be x or $x+1$. It is x when $x+y \not> 8$, except when the third figure is 0, which only occurs when the required number is 90. When there is no third figure from the end, the required number consists of y only, and does not exceed 8.

Consequently, we are able to give the following rule for instantly declaring the number:—Sum $a-b+3(c-d)$ and add the resulting digits together. If positive, deduct this from 9 and also from 18. If negative, add it to 0 and also to 9. Then we know that $x+y$ = one or other of the two numbers arrived at, and e shows which, being itself x or $x+1$, as shown above. The finding of y is then easy. But, unless the memory be exceptionally good, it is advisable to jot down a, b, c, d as they are declared, putting them in the form $a-b+3(c-d)$, and finding the two possible values of $(x+y)$ in readiness to give the correct answer immediately on the declaration of e. All this can be done by A. whilst B. is engaged in his work.

[Professor BOURNE gives the Solution as follows:—

Let the numbers tacked on be t_1 and t_2, and the respective remainders be r_1 and r_2; subtract r_1 from t_1 and to the result add, if necessary, 10 or 20, so as to make the result a multiple of 3; call this $3p$. Similarly, from t_2 and r_2 obtain $3q$.

Take the digit in the tens' place of $3q$ and subtract it from p, adding 10 or 20 if necessary, as before; call the result $3l$.

Let the figure third from the right in the final quotient be a; then the required original number is $3\,(3a + \text{ten's digit of } 3l) + \text{ten's digit of } 3p$.

The respective quantities a, l, q are the digits, read from the left, of the final quotient, and by working backwards from this, the reason of the rule is seen.

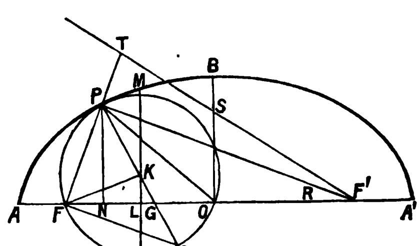
The work can be thus exhibited:—

If t_1 and t_2 are	9 and 8,
and r_1 and r_2 are	1 and 1,
then $3p$ and $3q$ are	18 and 27.
Also, p is	6.
Hence, $3l$ is	24.

If the value of a is 7, multiply 7 by 3, and add in the 2 of the 24; multiply the result by 3, and add in the 1 of the 18; the final result is 70.]

12674. (Professor Droz-Farny.)—Soient O le centre et F un des foyers d'une ellipse; trouver sur cette dernière un point P d'où le segment OF soit vu sous un angle maximum; construire géométriquement ce point.

Solution by H. W. Curjel, M.A.; D. Biddle; *and others.*



P is the point of contact of a circle which passes through OF and touches the ellipse. For the angles subtended by a given chord of any circle from all points on the circumference (on the same side of the chord) are equal, and exceed those subtended by the same chord from points outside the circle. The centre of the circle K is on the perpendicular LM, which bisects OF; and PK, being normal to the ellipse, bisects the angle FPF′. If PK be produced to Q, PFQ is a right angle; moreover, $\angle$ PQF = POF. Therefore, if PN be drawn perpendicular to AA′,

$$\angle \text{OPN} = \text{FPQ} = \tfrac{1}{2}\text{FPF}', \quad \text{and} \quad \angle \text{PON} = \tfrac{1}{2}(\text{PFO} + \text{PF}'\text{O}),$$

whilst $$\angle \text{NPQ} = \tfrac{1}{2}(\text{NPF}' - \text{NPF}) = \tfrac{1}{2}(\text{PFO} - \text{PF}'\text{O}).$$

$$\text{PN/ON} = \tan\tfrac{1}{2}(\text{PFO} + \text{PF}'\text{O}), \quad \text{and} \quad \text{NG/PN} = \tan\tfrac{1}{2}(\text{PFO} - \text{PF}'\text{O}).$$

Consequently, $$\text{PN/ON} : \text{NG/PN} = \text{PF}' + \text{PF} : \text{PF}' - \text{PF}.$$

We also have, by a well-known formula, NG : ON = OB² : OA², whence $\quad OB^2 . ON^2 : PN^2 = OA : OA - PF.$

Next we have $\quad (OF' + ON)^2 + PN^2 = PF'^2,$

whilst $(OF - ON)^2 + PN^2 = PF^2$, whence $PF' - OA = OF . ON$, and finally $\quad PF' = (OF'^2 + \frac{1}{4}OA'^2)^{\frac{1}{2}} + \frac{1}{2}OA'.$

Thus the construction is extremely simple. Make OS, on OB, $= OR = \frac{1}{2}OA'$. Join F′S and produce to T, making ST = OS. With centre F′ and radius F′T, describe the arc TP cutting the ellipse in the required point.

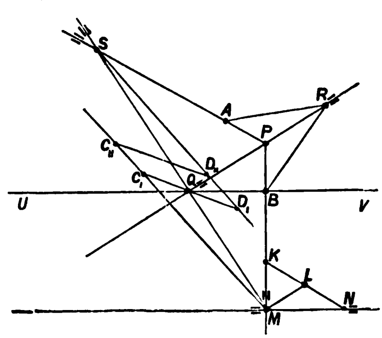
[The Proposer sends the following "construction du point P. Prolongeons le petit axe OB d'une longueur BE de manière à ce que OE = OA = a. Prolongeons de même le grand axe OA d'une longueur AH de manière à ce que OH = 2OF. EN sera la bissectrice de l'angle HEO." He remarks that it would be "intéressant de déduire géométriquement ce résultat de la belle solution de Mr. Curjel."

12396. (H. J. Woodall, A.R.C.S.)—Give a description of a suitable instrument for the continuous drawing of any conic section.

Solution by the Proposer.

At the meeting of the Physical Society held May 13th, 1892, Mr. Inwards exhibited an instrument for the continuous description of parabolas, the ultimate principle of which is that the curve is the locus of a point whose perpendicular distance from the directrix is equal to its distance from the focus.

To generalize the instrument, I took the general definition of a conic (similar to that above) and, by means of mechanisms for (a) dividing a straight line in a given ratio and (b) bisecting a given angle, obtained an instrument for the continuous description of any conic section.



In the figure we have S the focus, P the tracing point, MN the directrix. (a) Mechanism for dividing a straight line in a constant ratio

$$SD_{,,}D_{,}QC_{,,}C_{,}M,$$

where $D_{,,}D_{,}C_{,}C_{,,}$ is a jointed parallelogram, and

$$SD_{,} : C_{,}M = D_{,}Q : QC_{,};$$

whence Q divides SM in a ratio $= SD_{,} : C_{,}M$ = constant (this linkage may be left out if a bar as UV fixed across the paper, for Q to slide on, be sufficient). (b) Mechanism for bisecting an angle ARBP,

where AP = BP and AR = RB, and the bar jointed at P passes through R by a slide.

Now, by Euc. VI. 3, if PR passes through Q, we shall have

$$SP : PM = SQ : QM = \text{constant};$$

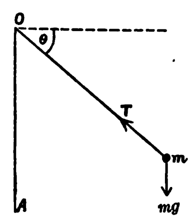
whence the construction.

It only remains to notice how PM is kept perpendicular to the directrix by KLMN, for, if LK = LM = LN, then KMN is a right angle.

[Mr. WOODALL states that this mechanism was described to his fellow-students at the Royal College of Science on May 14th, 1892. (A joint is indicated by a dot, a slide by two parallel lines about the sliding member.)]

12631. (A. S. EVE.)—A heavy particle is attached to an elastic string and rotates as a conical pendulum about a vertical axis. When different values are given to the angular velocity, prove that the particle traverses horizontal circular sections of the surface generated by rotating a conchoid of Nicomedes ($r = a + b \operatorname{cosec} \theta$) about the vertical axis; where a is the natural length of the string, and b is the extension due to the weight of the particle.

Solution by W. J. DOBBS, M.A.; Professor SANJÁNA; *and others.*



Let r be the length of the string when inclined at angle θ to the horizontal, m the mass of the particle. It is easily seen that the modulus of elasticity of the string is mga/b, so that, if T be the tension of the string in absolute units,

$$T = (mg/b)(r-a).$$

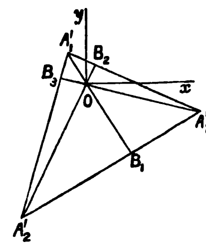
Resolving vertically, we have $T \sin\theta = mg$;

$$\therefore \quad r = a + b \operatorname{cosec}\theta,$$

which proves the theorem. [The other equation of motion goes to determine the angular velocity.]

12689. (A. C. DIXON.)—Three particles of given masses m_1, m_2, m_3 are to be placed in such a way as to have assigned axes and moments of inertia. Show that (1) the locus of each is an ellipse, similar, similarly situated, and concentric with the given momental ellipse; (2) the eccentric angles of the three particles on their respective ellipses differ by constants which depend only on the masses; (3) the envelope of each side of the triangle formed is another similar, similarly situated, and concentric ellipse, and the triangle is self-conjugate with respect to a fixed imaginary conic; (4) the area of the triangle is constant, and each side of it varies as the diameter parallel to the tangent at the opposite vertex; and (5) extend to the case of four particles with a given momental ellipsoid.

Solution by Professor Schoute; H. W. Curjel, M.A.; *and others.*



If (x_i, y_i) be the coordinates of the three points A_i with the masses m_i $(i = 1, 2, 3)$ with reference to the given axes of inertia, and M, Ma^2, Mb^2 represent the sum and the two moments of inertia of the masses, the five conditions of the problem are

$$\Sigma m_i x_i = \Sigma m_i y_i = \Sigma m_i x_i y_i = 0,$$
$$\Sigma m_i x_i^2 = Ma^2, \quad \Sigma m_i y_i^2 = Mb^2.$$

A rather troublesome elimination of the coordinates of two of the three points leads to the equation of the locus of the third one, viz.,

$$m_i (b^2 x_i^2 + a^2 y_i^2) = (M - m)\, a^2 b^2.$$

But this elimination can be avoided altogether. By the substitution $ay_i = by_i'$, we alight on the new points A_i' with the coordinates (x_i, y_i'), that in bearing the masses m_i give rise to a circle of inertia. Now, we first determine the different positions of the triangle A_i', and then go back to the original one, A_i.

In the case of the triangle A_i', the five equations

$$\Sigma m_i x_i = \Sigma m_i y_i' = \Sigma m_i x_i y_i' = 0, \quad \Sigma m_i x_i^2 = \Sigma m_i y'^2 = Ma^2$$

hold with respect to any pair of rectangular axes through the centre of gravity O. If OA_i' be a new axis of x, the first and the third equation show that the side opposite to A_i' is parallel to the new axis of y. In other terms, the centre of gravity O of the masses m_i is, at the same time, the orthocentre of the triangle. Moreover, the two equations

$$m_i A_i'O = (M - m_i)\, B_i O, \quad m_i A_i'O^2 + (M - m_i)\, B_i O^2 = Ma^2$$

give $\quad A_i'O = a\left\{(M - m_i)/m_i\right\}^{\frac{1}{2}}, \quad B_i O = a\left\{m_i/(M - m_i)\right\}^{\frac{1}{2}}.$

These relations prove the triangle A_i' to be of a quite determinate form and size; therefore (1) the loci of the points A_i are circles, centre C, radii $a\left\{(M - m_i)/m_i\right\}^{\frac{1}{2}}$; (2) the angles A_i', O, A_{i+1}' are constant; (3) the envelopes of the sides (opposite to A_i') are circles, centre C, radii $a\left\{m_i/(M - m_i)\right\}^{\frac{1}{2}}$, and the triangle A_i' is self-conjugate with respect to the imaginary circle, centre O, radius $a(-1)^{\frac{1}{2}}$ (for $A_i'O \,.\, B_i O = a^2$); (4) the area of the triangle A_i' is constant, and its sides are constant too.

By returning to the points A_i, the assertions of the problem are proved.

(5) The extension of the problem to three and more dimensional space admits the same geometrical treatment. If we are given

$$\Sigma m_i x_i = 0, \; \Sigma m_i y_i = 0, \; \Sigma m_i z_i = 0, \; \Sigma m_i y_i z_i = 0, \; \Sigma m_i z_i x_i = 0, \; \Sigma m_i x_i y_i = 0,$$
$$\Sigma m_i x_i^2 = M'a^2, \quad \Sigma m_i y_i^2 = M'b^2, \quad \Sigma m_i z_i^2 = M'c^2,$$

where M′ means $m_1 + m_2 + m_3 + m_4$, we put $ay_i = by_i'$, $az_i = cz_i'$, and prove in the same way that the centre of gravity O of the four masses m_i is the

orthocentre of the tetrahedron $A_1'A_2'A_3'A_4'$, that this tetrahedron has an invariable form and size, and that the relations

$$A_i'O = a\left\{(M'-m_i)/m_i\right\}^{\frac{1}{2}}, \quad B_iO = a\left\{m_i/(M'-m_i)\right\}^{\frac{1}{2}}$$

hold. So we find four theorems that are the evident extensions of the four enumerated ones.

In this case the number of conditions is nine. Therefore it seems impossible to alight on the equation of the locus of A_i' by eliminating the nine coordinates of the three other points. And in space of four dimensions the fourteen equations must be of such a nature that the elimination of sixteen coordinates leads to the equation of the hyperspheric locus of A_i', &c.

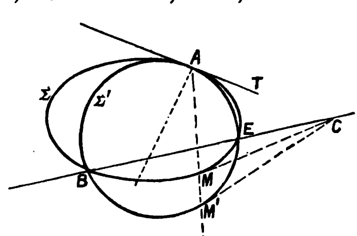
[The Proposer remarks that the elimination is easily accomplished if we notice that $x_1/a.(m_1/M)^{\frac{1}{2}}$, $x_2/a.(m_2/M)^{\frac{1}{2}}$, $x_3/a.(m_3/M)^{\frac{1}{2}}$; $y_1/b.(m_1/M)^{\frac{1}{2}}$, $y_2/b.(m_2/M)^{\frac{1}{2}}$, $y_3/b.(m_3/M)^{\frac{1}{2}}$; $(m_1/M)^{\frac{1}{2}}$, $(m_2/M)^{\frac{1}{2}}$, $(m_3/M)^{\frac{1}{2}}$, are the direction-cosines of a system of rectangular axes. We can then write down at once such equations as

$$m_1/M.(x_1^2/a^2+y_1^2/b^2+1) = 1, \quad (m_2m_3)^{\frac{1}{2}}/M.(x_2y_3-x_3y_2)/ab = \pm(m_1/M)^{\frac{1}{2}},$$

from which the assertions of the problem follow. A like method may be used for (6).]

12553. (Professor Verbessern.)—Soit AT la tangente en un point donné A d'une conique. La tangente en un point variable M de la conique rencontre la symétrique de AT par rapport à la corde AM en un point P. Démontrer que le point P décrit une droite.

Solution by Professor Droz-Farny; H. W. Curjel, M.A.; *and others.*



Soit Σ' un cercle quelconque tangent en A à la conique donnée Σ. Les deux courbes ont, outre la tangente en A, la corde BE en commun.

Une corde variable quelconque par A rencontre Σ en M et Σ' en M' et on sait que les tangentes en M et M' se rencontrent en un point C de la corde commune BE; or les tangentes AT et CM' sont isoscéliennes par rapport à AM.

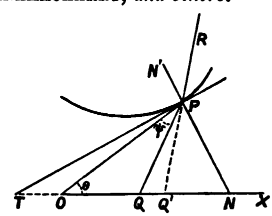
Supposons le cercle Σ' infiniment petit, la tangente M'C deviendra la symétrique de AT par rapport à la corde AM et le point C deviendra le point P. Le lieu cherché est donc la corde d'intersection de la conique Σ et du cercle point A. Elle passe évidemment par le pôle de la normale en A et son pôle coïncide avec le point de Frégier de A.

3078. (Professor Hudson, M.A.)—A ray of light traverses a medium in which the density at any point is a function of (r, θ), the polar coordinates of the point; prove that, if μ be the refractive index,

$$\frac{\mu}{\rho} = \frac{\cos\psi}{r}\frac{d\mu}{d\theta} - \sin\psi\frac{d\mu}{dr},$$

where ρ is the radius of curvature of the path of the ray, and ψ the inclination of its tangent to the radius vector.

Solution by Professors Sanjána, Krishnamacharry, *and others.*



Constructing as above (Quest. 3102), we have OP $= r$, $\angle$ POQ $= \theta$, $\angle$ OPQ $= \psi$.

The intrinsic equation of the path of the ray is $\frac{\mu}{\rho} = \frac{d\mu}{dy}\frac{dx}{ds} - \frac{d\mu}{dx}\frac{dy}{ds}$ (see Heath's *Optics*, p. 335).

Now, $\cos(\theta+\psi) = dx/ds$,
$\sin(\theta+\psi) = dy/ds$.

Also $$\frac{d\mu}{dx} = \frac{x}{r}\frac{d\mu}{dr} - \frac{y}{r^2}\frac{d\mu}{d\theta} = \cos\theta\frac{d\mu}{dr} - \frac{1}{r}\sin\theta\frac{d\mu}{d\theta},$$

and $$\frac{d\mu}{dy} = \frac{y}{r}\frac{d\mu}{dr} + \frac{x}{r^2}\frac{d\mu}{d\theta} = \sin\theta\frac{d\mu}{dr} + \frac{1}{r}\cos\theta\frac{d\mu}{d\theta}$$

(see Todhunter's *Differential Calculus*, p. 181).

By substitution we obtain $\frac{\mu}{\rho} = -\frac{d\mu}{dr}\sin\psi + \frac{1}{r}\frac{d\mu}{d\theta}\cos\psi$.

12697. (R. W. D. Christie.) — Is $N = a^2+b^2 = c^2+d^2 = e^2+f^2$ possible in integers; *i.e.*, can an integer be resolved into a pair of square integers in more than *two* ways?

Solution by Professor Mathews; R. W. Hogg, M.A.; *and others.*

This is a well-known question in the Theory of Numbers (see, for instance, Professor Mathews' book, pp. 36, 97, 98, or Smith's *Report*, Art. 95). For example, let $m = 5 \cdot 13 \cdot 17 = 1105$;

then $$m = 961+144 = 31^2+12^2 = 1089+16 = 33^2+4^2$$
$$= 1024+81 = 32^2+9^2.$$

Numbers can be found which are decomposable into the sum of two squares in any required number of ways.

[Professor Mathews thinks that people interested in the Theory of Numbers should not persistently refuse to make any effort to find out

what is known about the subject. This question was settled by FERMAT more than 200 years ago, and has been referred to in every text-book on the subject.

Dr. ARTEMAS MARTIN remarks that if the PROPOSER thinks that no number can be the sum of two integral squares in more than *two* ways he is certainly mistaken; and he refers to EULER's *Commentationes Arithmeticæ Collectæ*, Vol. I., p. 157, where we find $1105 = 33^2+4^2 = 32^2+9^2 = 31^2+12^2 = 24^2+23^2$.]

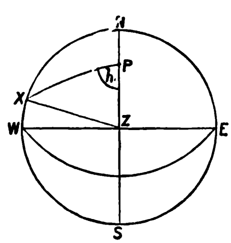
12691. (W. J. DOBBS, M.A.)—A, B, C are three points in the plane of a circle on which are taken any two points 1 and 2. 1A and 2A meet the circle again in a_1 and a_2 respectively, and similarly for the points B and C. Prove that (1) BC, b_1c_2, b_2c_1 are concurrent in P, (2) CA, c_1a_2, c_2a_1 are concurrent in Q, (3) AB, a_1b_2, a_2b_1 are concurrent in R, (4) P, Q, R are collinear.

Solution by Professor SCHOUTE; H. W. CURJEL, M.A.; *and others.*

The problem is solved by the application of PASCAL's theorem on the four hexagons $(1b_1c_2,\ 2b_2c_1)$, $(1c_1a_2,\ 2c_2a_1)$, $(1a_1b_2,\ 2a_2b_1)$, $(a_1b_2c_1,\ a_2b_1c_2)$. For the circle may be substituted the most general conic.

12659. (C. BICKERDIKE.)—Required the latitude of the place and the declination of the sun, when the length of the day is to that of the night as 3 to 2; and the sun's meridian altitude to his depression at midnight is as 2 to 1.

Solution by C. MORGAN, R.N.; Prof. BHATTACHARYA; *and others.*



Let a = meridian altitude,

b = depression at midnight,

d = declination, l = latitude of place;

$\cos h = -\tan d \tan l$; $h/(180-h) = \frac{3}{2}$;

$\therefore\ h = 108°$.

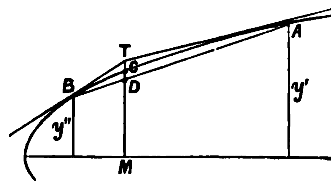
Also, polar distance at noon = polar distance at midnight (approximately).

$\therefore\ 90-d = 90-l+90-a = l+b$, and $a = 2b$,

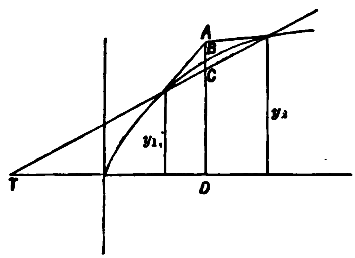
$\therefore\ l = 90-3d$;

$$\text{whence}\quad \sin 18 = \frac{1-3\tan^2 d}{3-\tan^2 d},\quad \left.\begin{array}{l} d = 9°\ 20' \\ l = 62° \end{array}\right\}.$$

12662. (C. H. Oldham, B.A., B.L.)—A and B are any two points of a parabola, the tangents at which intersect at T. A perpendicular is dropped from T to meet the principal axis of the parabola at M; and this perpendicular TM meets the parabola at the point C, and the chord AB at the point D. Show that the lengths TM, CM, and DM are, respectively, the arithmetic mean, the geometric mean, and the harmonic mean of the perpendiculars from A and B to the principal axis.



Solution by R. Chartres; Professor Sanjána; *and others.*



The coordinates of A are

$$y_1y_2/4a, \quad \tfrac{1}{2}(y_1+y_2),$$

and $\quad CD = TD \tan T$

$$= (2y_1y_2)/(y_1+y_2).$$

Hence AD, BD, CD are the arithmetic, geometric, and harmonic means between y_1 and y_2.

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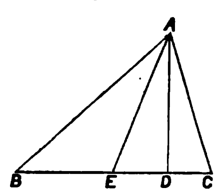
619. (A. D. Balfour.)—To find a triangle such that the three sides, perpendicular, and line bisecting the base from the opposite angle may all be expressed in rational numbers.

Solution by Professors Zerr, Aiyar, *and others.*



Here $AD^2 = AB^2 - BD^2 = AC^2 - DC^2$,

$$AD^2 + \tfrac{1}{4}(BD-DC)^2 = AE^2.$$

Let $\quad AD = 12x, \quad AB = 20x, \quad AC = 15x,$

$$BD = 16x, \quad DC = 9x.$$

Then $(12x)^2 = (20x)^2-(16x)^2 = (15x)^2-(9x)^2$,

$$(12x)^2 + \tfrac{1}{4}(16x-9x)^2 = \tfrac{625}{4}x^2 = AE^2,$$

$$\therefore \quad AE = \tfrac{25}{2}x.$$

Hence,

$$AD = 24x, \quad AB = 40x, \quad AC = 30x, \quad BC = BD + DC = 50x, \quad AE = 25x,$$

and from this we can get any number of triangles by giving different values to x; when $x = 1$ we get the least values.

12199. (Professor LESTING.)—On donne, sur une même droite, trois points équidistants A, B, C. De A comme centre avec AB pour rayon, on décrit un cercle; sur une tangente quelconque à ce cercle, on abaisse la perpendiculaire CD, et l'on tire BD. Démontrer que l'angle ABD est le triple de l'angle BDC; et, en outre, chercher le lieu du point D.

Solution by R. CHARTRES; R. H. W. WHAPHAM; *and others.*

$$\angle \text{BDC} = \text{BFC} = 90 - \tfrac{1}{2}\text{A},$$

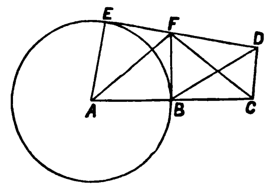
and $\quad \text{ABD} = 180 - \text{A} + 90 - \tfrac{1}{2}\text{A};$

therefore $\quad \text{ABD} = 3\text{BDC}.$

If $\quad \text{BD} = r,$

$\quad \text{ABD} = \theta,$

we have $\quad r = 2a\cos\tfrac{1}{3}\theta.$



12483. (J. J. BARNIVILLE, B.A.)—Prove that

$$\frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{14^2} + \dots = \frac{4\pi^2 - 31}{27},$$

$$\frac{2}{1^3 + 2^3} + \frac{3}{3^3 + 4^3 + 5^3} + \frac{4}{6^3 + 7^3 + 8^3 + 9^3} + \dots = 2\pi^2 - 19\tfrac{1}{2}.$$

Solution by Professor SANJÁNA; H. W. CURJEL, M.A.; *and others.*

Let u_n, v_n be the nth terms of the series. Then

$$u_n = \frac{4}{n^2(n+3)^2} = -\tfrac{8}{9}\left\{\frac{1}{n(n+1)} - \frac{2}{n(n+1)(n+2)} + \frac{2}{n(n+1)(n+2)(n+3)}\right\} + \tfrac{4}{9}\left\{\frac{1}{n^2} + \frac{1}{(n+3)^2}\right\};$$

$$\therefore\ \Sigma_1^\infty u_n = -\tfrac{8}{9}\left\{1 - \frac{2}{4} + \frac{2}{18}\right\} + \tfrac{4}{9}\left\{\frac{\pi^2}{6} + \frac{\pi^2}{6} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^3}\right\} = \frac{4\pi^2 - 31}{27},$$

$$v_n = \frac{8}{n^2(n+1)^2(n+2)^2} = \frac{2}{n^2} + \frac{8}{(n+1)^2} + \frac{2}{(n+2)^2} - \frac{12}{n(n+1)} + \frac{12}{n(n+1)(n+2)}.$$

$$\Sigma_1^\infty v_n = \frac{12\pi^2}{6} - 8 - 2\left(\frac{1}{1^2} + \frac{1}{2^2}\right) - 12 + 3 = 2\pi^2 - 19\tfrac{1}{2}.$$

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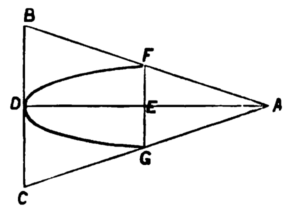
505. (R. G. CLARKE.)—Within a given isosceles triangle inscribe the greatest parabola, the vertex of the latter being at the middle of the base of the triangle.

Solution by Professors Zerr, Bhattacharya, *and others.*



Let $AD = a$ = altitude of triangle,
$BC = b$ = base ,, ,,
$DE = x$ = altitude of parabola,
$FG = y$ = base ,, ,,

then $\frac{2}{3}xy$ = maximum;

but $a-x : a = y : b$ or $y = a/b(a-x)$;

$\therefore\ x(a-x)$ = maximum;

taking the differential, we get

$$x = \tfrac{1}{2}a, \quad y = \tfrac{1}{2}b;$$

$$\therefore\ \tfrac{2}{3}xy = \tfrac{1}{6}ab = \tfrac{1}{3}\text{ (area of triangle).}$$

12492. (Rev. T. C. Simmons, M.A.)—Three dice are to be thrown once. If a doublet appears, A is to receive the square of the numbers turned up plus their sum; if a triplet appears, B is to receive the cube of the numbers turned up plus their sum. The expectation of A is exactly equal to that of B. Can any reason be given for this, without going through the laborious process of a detailed calculation? [See Vol. xliv., p. 42, where the calculation is given. The agreement of the two results is a mystery to the Proposer, who is inclined to treat it as a remarkable accidental coincidence. We shall be glad if some correspondent will point out a *law* connecting the two expectations.]

Solution by H. Fortey.

Let the data be exactly as in the question, except that each die has n faces marked from 1 to n. Call A's expectation E_2, and B's, E_3.

Then $E_2 = \dfrac{n+1}{n^2}(n-1)(8n+9)$, $E_3 = \dfrac{3(n+1)}{4n^2}(3n+1)(3n+2)$;

therefore $$\frac{E_2}{E_3} = \frac{4(n-1)(8n+9)}{3(3n+1)(3n+2)} = \rho, \text{ suppose.}$$

The value of ρ increases from $\rho = \frac{25}{42}$, when $n = 2$, to $\rho = \frac{32}{27}$, when $n = \infty$, and when $n = 6$, $\rho = 1$. It is, of course, a curious coincidence that the conditions of the question should make $\rho = 1$, when ordinary dice are used.

9820, 9821, 9808. (By W. J. C. Sharp, M.A.) — 9820. Prove the following—(i.) If abc and $A'B'C'$ be the pedal triangles of the circumcentre O of the triangle ABC and of any other point P, $A'B'C' = \frac{1}{4}\Delta ABC \times (R^2 \sim OP^2)/R^2$, where R is the circumradius. (ii.) If O, K be the centres of two circles whose radii are R, r, P any point on the second circle, and PL the perpendicular from P to the radical axis of the circles, $2OK.PL = R^2 \sim OP^2$. (iii.) The area of the pedal triangle of any point

P on a circle, the centre of which is K, with respect to a triangle ABC, of which O is the circumcentre and R the circumradius, is $\frac{1}{4}\Delta ABC\,(OK.PL)/R^2$, where PL is the perpendicular from P upon the radical axis of the two circles.

(9821) Show that, if a point be taken at random in the circumscribed circle of a triangle, the mean area of the pedal is $\frac{1}{8}$ of the triangle.

(9808) The mean value of the pedal triangle of a random point in a triangle ABC is $\frac{2}{3}R^2(1+\cos A\cos B\cos C)$, where A, B, C are the angles of the triangle, and R the circumradius.

Solution by H. J. WOODALL, A.R.C.S.

9820. (i.) If the coordinates of P be (α, β, γ), then the area of the pedal triangle $= \frac{1}{2}(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C) =$ constant (H) if (α, β, γ) lies on the circle

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 2a\mathrm{H}\,(a\alpha+b\beta+c\gamma)^2/\left\{\sin A\,(2\Delta)^2\right\}$$
$$= \mathrm{K}\,(a\alpha+b\beta+c\gamma)^2.$$

This has double contact on line infinity (and is therefore concentric) with circumcircle.

We can easily show that the maximum value of H is when P is $(R\cos A,\ R\cos B,\ R\cos C)$, *i.e.*, the circumcentre, in which case $\mathrm{H} = \frac{1}{4}\Delta$.

The area of the pedal triangle of a point at a distance k from the circumcentre is (*vide supra*) dependent only on k; to find it, substitute for

(α, β, γ) $\quad \left\{(R\cos A - k)\ (R\cos B + k\cos C)\ (R\cos C + k\cos B)\right\}$;

we get $\quad 2\mathrm{H} = (R^2-k^2)\sin A\sin B\sin C$;

$$\therefore\quad \mathrm{H} = \tfrac{1}{2}(R^2-k^2)\sin A\sin B\sin C = \tfrac{1}{4}\Delta ABC \times (R^2 \sim OP^2)/R^2.$$

(ii.) This is a case of the theorem proved (CASEY, "Sequel," Book III., Prop. 24), viz., when P is on the circle whose centre is K.

$$\therefore\quad 2OK.PL = R^2 \sim OP^2.$$

(iii.) Applying this to the result in (i.), we get

$$\mathrm{H} = \tfrac{1}{2}\Delta ABC\,(OK.PL)\,R^2.$$

9821. From (i.) of 9820 we have $4A'B'C'/ABC = (R^2 \sim OP^2)/R^2$.
The mean of this result $= \frac{1}{4}R^4/\frac{1}{2}R^4 = \frac{1}{2}$.
Therefore mean area of $A'B'C' = \frac{1}{8}ABC$.

9808. From (i.) of 9820 we have

$$\mathrm{H} = \tfrac{1}{2}(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C);$$

substituting for γ from $\quad a\alpha + b\beta + c\gamma = 2\Delta$,
and then taking the average by means of the formula average

$$= \iint \mathrm{H}\,d\beta\,d\alpha \Big/ \iint d\beta\,d\alpha,$$

we get that average to be $= \frac{1}{3}R^2\sin A\sin B\sin C\,(1+\cos A\cos B\cos C)$.

12261. (Professor LAMPE.)—Let P be a point of the ellipse $x^2/a^2+y^2/b^2=1$. There are three normals PF_1, PF_2, PF_3, distinct from the normal at P, which may be drawn to the points F_1, F_2, F_3 of the ellipse. Prove that the locus (1) of the centroid of the triangle $F_1F_2F_3$ is a similar and concentric ellipse with the semi-axes $\frac{1}{3}a\dfrac{a^2+b^2}{a^2-b^2}$, $\frac{1}{3}b\dfrac{a^2+b^2}{a^2-b^2}$; and (2) explain the curious fact that this locus, whose points ought to lie within the ellipse, coincides with the ellipse if $a=b\sqrt{2}$, and is exterior to the ellipse if $a<b\sqrt{2}$.

Solution by the PROPOSER, Professor CHAKRIVARTI, *and others.*

(1) Let x_1, y_1 be the coordinates of a point of the ellipse; then the equation of the normal at (x_1, y_1) will be $a^2xy_1-b^2x_1y=(a^2-b^2)\,x_1y_1$ ... (1).

If P has the coordinates x, y, and the foot of the normal from P to the ellipse the coordinates x_1, y_1, we may reduce (1) to x and x_1. Putting $a^2-b^2=c^2$, we get

$$c^4x_1^4-2a^2c^2xx_1^3+x_1^2\left\{a^2(b^4-c^4)+x^2(a^4-b^4)\right\}+2a^2c^4xx_1-a^6x^2=0.$$

Dividing by x_1-x, and writing ϵ for c/a, this becomes

$$\epsilon^4x_1^3+(\epsilon^4-2\epsilon^2)\,xx_1^2+a^2(1-2\epsilon^2)\,x_1+a^2x=0 \quad \text{............ (2).}$$

Proceeding in the same way with y, we get

$$\epsilon^4y_1^3+(2\epsilon^2-\epsilon^4)\,yy_1^2+a^2(1-\epsilon^2)(1-\epsilon^4)\,y_1+(1-\epsilon^2)^3a^2y=0 \quad \text{... (3).}$$

The coordinates of the centroid of $F_1F_2F_3$ are, from (1, 2),

$$\xi=\tfrac{1}{3}(x_1+x_2+x_3)=\tfrac{1}{3}x\,(2/\epsilon^2-1), \qquad \eta=\tfrac{1}{3}(y_1+y_2+y_3)=\tfrac{1}{3}y\,(1-2/\epsilon^2).$$

But $\quad x^2/a^2+y^2/b^2=1; \qquad \therefore\ 9\xi^2/a^2+9\eta^2/b^2=(2/\epsilon^2-1)^2,$

the equation of an ellipse with the semi-axes

$$\tfrac{1}{3}a\,(2/\epsilon^2-1)=\tfrac{1}{3}a\,\frac{a^2+b^2}{a^2-b^2},\quad \tfrac{1}{3}b\,\frac{a^2+b^2}{a^2-b^2}.$$

(2) Four real normals are possible from points lying inside the evolute. If the evolute does not cut the ellipse (which happens to be for $a<b\sqrt{2}$), there is only one real normal among the above three normals; in this case the triangle $F_1F_2F_3$ has two imaginary vertices, but its centroid remains real, passing outside the ellipse.

12129. (Professor MANDART.)—Quelles sont les courbes dont le cercle de courbure passe par un point fixe?

Solution by H. W. CURJEL, B.A.; Professor SARKAR; *and others.*

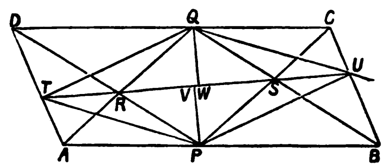
Take the fixed point O as origin and let C be the centre of curvature; $OC^2=r^2+\rho^2-2p\rho$. Since circle of curvature passes through O,

$$r^2=2p\rho=2rp\,dr/dp\,; \quad \therefore\ dp/p=2dr/r\,; \quad \therefore\ r^2=ap\,;$$

therefore curve is a circle.

12711. (Professor ANTHONY.)—From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertices. Prove that a straight line through the points of intersection of these lines bisects the parallelogram.

Solution by M. BRIERLEY; C. BICKERDIKE; *and others.*



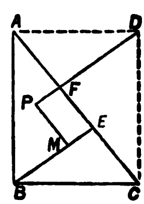
Let ABCD be the given parallelogram, P, Q the given points, R, S the points of intersection of the cross lines to the vertices, and TU the dividing line; also PV, QW perpendiculars upon TU. The triangles DPC, AQB are equal to one another; so are RQS and RPS. Hence RS . PV = RS . QW and PV = QW;

$\therefore$ TU . PV = TU . QW = $\frac{1}{4}$ parallelogram ABCD;

$\therefore$ ATUB = $\frac{1}{2}$ parallelogram ABCD = PTQU.

963. (EDITOR.)—A card is folded in two along a diagonal; find the distance apart of the other two corners when the two parts of the card contain a given angle.

Solution by Profs. SANJÁNA, KRISHMACHANDRA DE, *and others.*



Let ABCD be the card, AB = a, BC = b. Let the perpendicular BE or DF be denoted by p, and EF by q.

Then $p = ab/(a^2+b^2)^{\frac{1}{2}}$, $q = (a^2-b^2)/(a^2+b^2)^{\frac{1}{2}}$.

If CAD makes an angle α with, and P is the projection of D on, the plane of ABC, the square of the required distance = DP² + PM² + BM²

$= (p\sin\alpha)^2 + q^2 + (p - p\cos\alpha)^2 = q^2 + 2p^2 - 2p^2\cos\alpha$

$= \left\{(a^2-b^2)^2 + 2a^2b^2 - 2a^2b^2\cos\alpha\right\}/\quad + b^2)$

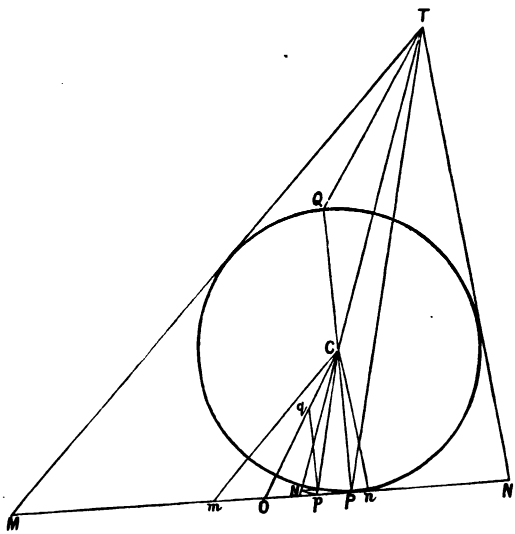
$= (a^4 + b^4 - 2a^2b^2\cos\alpha)/(a^2+b^2)$.

12626. (R. KNOWLES, B.A.)—From a fixed point T (hk) tangents are drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$; a variable tangent at P meets these in M and N; prove that (1) the locus of O, the mid-point of MN, is the hyperbola $(k^2-b^2)\,x^2 - 2hkxy + (h^2-a^2)\,y^2 + a^2b^2 = 0$; (2) if C be the centre of the ellipse, and Q the end of the diameter through P, CO is parallel to QT.

Solution by W. J. Dobbs, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

The following is a purely geometrical solution to this question :—

Let the vertical angle MTN and the inscribed circle of the triangle MTN be given in position while MN is a variable straight line. It is required to find the locus of O, the mid-point of MN.



In the figure, Cm and Cn are drawn parallel to TM and TN.

Then $\mathrm{M}m = m\mathrm{C}$ and $\mathrm{N}n = n\mathrm{C}$; $\therefore$ $\mathrm{C}m + m\mathrm{O} = \mathrm{C}n + n\mathrm{O}$;

hence $m\mathrm{O} = pn$, where p is the point of contact of mn with the inscribed circle of the triangle Cmn.

We proceed to find the locus of p. Draw pN perpendicular on TC.

Then CNpP is a cyclic quadrilateral so that $\angle p\mathrm{NP} = p\mathrm{CP} = \mathrm{CPT}$ and $\angle \mathrm{NP}p = \mathrm{NC}p = \mathrm{CTP}$. Thus triangle N$p$P is similar to triangle PCT; therefore $p\mathrm{P} : p\mathrm{N} = \mathrm{CT} : \mathrm{CP}$, a constant ratio; therefore the locus of p is a conic with centre C, to which it is easily seen that Cm and Cn are asymptotes, and the circle centre C its auxiliary circle.

Also, as $m\mathrm{O} = pn$, O is a point on the same conic. Also CO passes through q, the other end of the diameter pq, to the in-circle of $m\mathrm{C}n$.

[This is the converse of the following geometrical problem :—PQ is any diameter of a circle which touches two straight lines OA, OB. Show that, if any straight line be drawn perpendicular to PQ, the part of it intercepted between OA and OP is equal to the part of it intercepted between OB and OQ.]

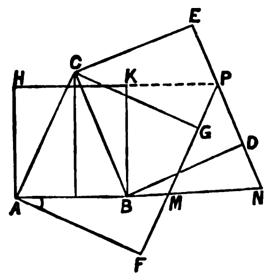
Therefore CO is parallel to QT. Now project the figure orthogonally, and we have the theorem enunciated.

12671. (Professor SANJANA, M.A. Extension of Quest. 12508.)—In a triangle ABC, if $\tan A = 2$, prove that (1)

$$b\sqrt{5}/(c+a\operatorname{cosec}B)+c\sqrt{5}/(b+a\operatorname{cosec}C) = 2;$$

and hence or otherwise, (2) if in such a triangle the squares ABKH and CAFG are drawn internally, and the square BCED externally, the lines DE, FG, HK are concurrent.

Solution by Professors DROZ-FARNY, RADAKUSHNAN, B.A., *and others.*



(1) Si $\tan A = 2$, $\sin A = 2/\sqrt{5}$, $\cos A = 1/\sqrt{5}$.

La relation à démontrer peut s'écrire

$$\frac{b\sin B}{a+c\sin B}+\frac{c\sin C}{a+b\sin C}=\frac{2}{\sqrt{5}}.$$

Or

$$\frac{b\sin B}{a+c\sin B}+\frac{c\sin C}{a+b\sin C} = \frac{b\sin B+c\sin C}{a+h} = \frac{b^2+c^2}{2aR+2hR} = \frac{(b^2+c^2)\sin A}{a^2+bc\sin A} = \sin A = \frac{2}{\sqrt{5}}$$

[h being the perpendicular from A on BC].

(2) Dans un triangle quelconque, les droites DE, FG, HK déterminent un triangle PQR semblable à ABC et dont la surface est égale à

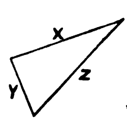
$$S(1-2\cot A)^2.$$

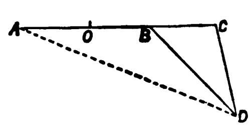
Si donc $\tan A = 2$, triangle PQR = 0, d'où le théorème.

12427. (H. D. DRURY, M.A.)—Produce a line AB to C, so that AC, CB may be equal to the square on a given line X; then, if we construct a triangle whose sides are AB, BC, X, the angle opposite the side X will be double the angle opposite the side BC. Hence show how Euc. IV. 10 follows.

Solution by Professors SANJÁNA, M.A., SARKAR, *and others.*

Bisect AB in O, take Y at right angles to X and equal to BO, and join the extremities of X and Y by Z; produce AB to C, making OC = Z.





Then, by Euc. II. 6 and I. 47, rectangle AC, CB = square on X. On

BC make the triangle BDC, so that BD = AB, CD = X. As X^2 = AC . CB, CD^2 = AC . CB; therefore CD touches the circumcircle of ABD.

Hence $\angle$BDC = BAD, Euc. III. 32, and BD = AB; therefore

$$\angle CBD = 2BDC.$$

Euc. IV. 10 will follow, by taking CD or X = AB or BD.

12715. (Professor Krishmachandra De, M.A.) — AA′ is a given finite straight line. AT and A′T′ are drawn at right angles to AA′ at its extremities. If AT and A′T′ be cut off from these perpendiculars, such that the rectangle contained by AT and A′T′ is constant ($= k^2$), prove that (1) the envelope of TT′ is an ellipse, when both T and T′ are taken on the same side of the straight line AP′, and determine its foci, distinguishing the cases when k is greater or less than $\frac{1}{2}$AA′; and (2) the envelope of TT′ is an hyperbola when T and T′ are taken on the opposite sides of AA′.

Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

If $k = \frac{1}{2}$AA′, the envelope is evidently a circle on AA′ as diameter. Hence it is evident that, if $k \neq \frac{1}{2}$AA′, the envelope is the orthogonal projection of a circle, one of the principal axes of which is AA′, and the other BB′ is the line joining the middle points of the two positions of TT′ when AT = A′T′.

The foci are on AA′, BB′, according as k is $<$ or $>$ $\frac{1}{2}$AA′ at a distance from the middle point of AA′ $= (\frac{1}{4}\text{AA}'^2 \sim k^2)^{\frac{1}{2}}$.

(2) Similarly, when AT, A′T′ are measured in opposite directions, *i.e.*, when k^2 is negative, the envelope is an hyperbola.

12672. (Professor Lampe, LL.D.)—Let P be a point of an ellipse; PF_1, PF_2, PF_3 the three normals, distinct from the normal at P, which may be drawn to the points F_1, F_2, F_3 of the ellipse; R the radius of curvature at P. Prove that $PF_1.PF_2.PF_3 = \dfrac{2a^2b^2R}{a^2-b^2}$, or the product of the three normals that may be drawn from a point on the ellipse to this curve varies as the radius of curvature at the point considered.

Solutions by Professors Radhakushnan, Chakrivarti, *and others.*

Taking the origin at P, and the coordinate axes parallel to the axes of the ellipse, let the equation of the ellipse be

$$ax^2 + by^2 + 2gx + 2fy = 0,$$

and let $y = mx$ be a normal from P.

Then, for the point where this cuts the ellipse,

$$x_1 = -2\frac{g+fm}{a+bm^2}, \quad y_1 = -2m\frac{g+fm}{a+bm^2}.$$

As the tangent at $(x_1 y_1)$ is perpendicular to $y = mx$, we get

$$by_1 + f = m(ax_1 + g);$$

or, substituting for x_1 and y_1, and reducing the result,

$$bm^3g - fm^2(2a-b) + mg(2b-a) - af = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

The three values of m correspond to the three normals.

Now $$\Sigma PF_1 = \Sigma(x_1^2+y_1^2)^{\frac{1}{2}} = -8\Sigma\frac{g+fm}{a+bm^2}(1+m^2)^{\frac{1}{2}}.$$

Transforming (1) into another variable n, such that $n = k + lm^2$, we get $$n_1n_2n_3 = \frac{kg^2(2bl-al-bk)^2 + lf^2(2ak-al-bk)^2}{b^2g^2};$$
from which, putting $k = a$ and $l = b$,

$$\Sigma a + bm^2 = \frac{4ab(a-b)^2(af^2+bg^2)}{b^2g^2}.$$

Similarly, putting $k = 1$ and $l = 1$,

$$\Sigma(1+m^2)^{\frac{1}{2}} = \left\{\frac{(a-b)^2(f^2+g^2)}{b^2g^2}\right\}^{\frac{1}{2}}.$$

Similarly, by another transformation, such that $n = fm + g$,

$$\Sigma fm + g = \frac{(bg^2+af^2)(f^2+g^2)}{bg};$$

hence we have $$PF_1 . PF_2 . PF_3 = \frac{8(f^2+g^2)^{\frac{3}{2}}}{4ab(a-b)} = \frac{2(f^2+g^2)^{\frac{3}{2}}}{ab(b-a)}.$$

Now, as $$[a]^2 = \frac{1}{a}\left(\frac{g^2}{a} + \frac{f^2}{b}\right), \quad [b]^2 = \frac{1}{b}\left(\frac{g^2}{a} + \frac{f^2}{b}\right),$$

and R at the origin by Newton's method $= +\dfrac{(g^2+f^2)^{\frac{3}{2}}}{af^2+bg^2}$,

$$\frac{2[a]^2[b]^2 R}{[a]^2 - [b]^2} = +\frac{2/(ab)\{(g^2/a)+(f^2/b)\}}{(1/a)-(1/b)}\frac{(g^2+f^2)^{\frac{3}{2}}}{af^2+bg^2}$$
$$= \frac{2}{ab(b-a)}(g^2+f^2)^{\frac{3}{2}} = PF_1 . PF_2 . PF_3.$$

[In this question Σ is used as a symbol for the continued product of the expressions of which the standard follows the symbol.

The Proposer remarks that this question was suggested to him by Quest. 3424. Designating the radius of curvature in the point of the parabola with R, the latus rectum with $2p$, the product of the two normals is found to be $= Rp$, an elegant result. Trying to extend this formula to the ellipse, he was led to the result enunciated in Quest. 12672. His method started from the central equation of the ellipse, $x^2/a^2 + y^2/b^2 = 1$. The three abscissæ of F_1, F_2, F_3 are the roots of the cubic equation

$$x^3 + \frac{\epsilon^2-2}{\epsilon^2}x_0 . x^2 + \frac{1-2\epsilon^2}{\epsilon^4}a^2x + \frac{a^2x_0}{\epsilon^4} = 0,$$

where x_0 is the abscissa of P, $\epsilon^2 = (a^2-b^2)/a^2$. Consequently the symmetric functions of its roots x_1, x_2, x_3 may be calculated from

$$x_1^2+x_2^2+x_3^2 = 1/\epsilon^4.\left\{(\epsilon^2-2)^2 x_0^2-2(1-2\epsilon^2)\,a^2\right\},$$

$$x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2 = 1/\epsilon^8.\left\{(1-2\epsilon^2)^2a^4-2a^2\epsilon^2(\epsilon^2-2)\,x_0^2\right\}.$$

The expression for PF_1^2 is moreover $PF_1^2 = \dfrac{4(1-\epsilon^2)(a^2-\epsilon^2x_1^2)^3}{\left\{a^2-\epsilon^2(2-\epsilon^2)\,x_1^2\right\}^2}$;

similarly for PF_1^2 and PF_3. Whence, working out the product of all three squares, we may reduce it to

$$PF_1.PF_2.PF_3{}^2 = \frac{2(1-\epsilon^2)^{\frac{1}{2}}(a^2-x_0^2\epsilon^2)^{\frac{3}{2}}}{\epsilon^2},$$

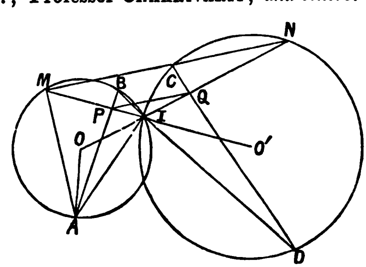
and having
$$R = \frac{(a^2-x_0^2\epsilon^2)^{\frac{3}{2}}}{a^2(1-\epsilon^2)^{\frac{1}{2}}},$$

we obtain $\quad PF_1.PF_2.PF_3 = \dfrac{2Ra^2(1-\epsilon^2)}{\epsilon^2} = \dfrac{2Ra^2b^2}{c^2} = \dfrac{2Ra^3p}{c^2}$,

where $c^2 = a^2-b^2$, $2p$ = latus rectum of the ellipse. Thus this demonstration resembles, in its order of conclusion, that given above. Comparing the formula for the ellipse with that for the parabola, we see that PF_3 is to be taken $= 2a^3/c\ (=\infty)$, if the two formulas are identified.]

12719. (Professor Duporcq.) — Deux cercles de centres O, O′ se coupent en I; on tire les droites O′I et OI, qui coupent respectivement les cercles O, O′ aux points M et N. Soit AB une corde quelconque du cercle O perpendiculaire en P à MO′; par le point Q où la parallèle à MN issue de P rencontre ON, on élève une perpendiculaire qui coupe le cercle O′ en deux points C, D. Démontrer que les quatre points A, B, C, D sont deux à deux en ligne droite avec le point I.

Solution by H. W. Curjel, M.A.; Professor Chakrivarti; *and others.*



Let B and C be on the smaller segments.

Then the segments MBI, ICN are similar; but

IP : PM = IQ : QN.

∴ the figures IQND, IPMA are similar.

∴ $\angle O'ID = \angle OIA$

$= \angle OAI = \angle PAM$

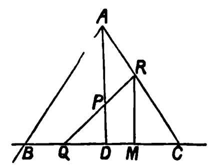
$= \angle MIB$.

∴ B, I, D are collinear.

Similarly, A, I, C are collinear.

12586. (EDITOR.)—If ABC be an isosceles triangle, AD a perpendicular to the base BC, AD = $\frac{8}{9}$BC, P a point in AD such that AP = $\frac{5}{8}$AD, and PQR a line cutting the sides in Q, R, and making $\angle$QPD = RPA = $\frac{1}{2}$ a right angle; prove (1) that QPR bisects the triangle, and (2) generalize the theorem.

Solution by Professors SANJÁNA, BHATTACHARYA, *and others.*



Draw RM perpendicular to BC.

Let AD = $8x$, BC = $9x$, RM = y; then

$$PD = QD = 3x, \quad RM = MQ;$$

also RM : MC = AD : DC; whence $y = \frac{24}{5}x$.

And $$RM \times QC = \tfrac{24}{5}x \times \tfrac{15}{2}x = 36x^2,$$
$$AD \times BC = 72x^2,$$

which proves the result.

One way of generalizing the theorem would be as follows:—

Let AD = px, BC = qx, PD = rx, and let PQR be drawn as before. Then RM = $y = \{p(2r+q)\}/(2p+q)$. If we put RM $\times$ QC = $\frac{1}{2}$AD $\times$ BC, we shall get $(2r+q)^2 = q(2p+q)$; whence $r = \frac{1}{2}\left[\{q(2p+q)\}^{\frac{1}{2}} - q\right]$. In the present case, $q = 9$, $p = 8$; therefore $r = 3$. When $q = 2$, $p = 3$, r will be 1; when $q = 8$, $p = 5$, r will be 2; and so on.

12645. (Professor MATZ.)—Find (1) nine positive integral numbers in arithmetical progression the sum of whose squares is a square number; and (2) nine integral square numbers whose sum is a square number.

Solution by Dr. ARTEMAS MARTIN; Rev. J. L. KITCHIN, M.A.; *and others.*

1. Let $x-4y$, $x-3y$, $x-2y$, $x-y$, x, $x+y$, $x+2y$, $x+3y$, $x+4y$ denote the required numbers in arithmetical progression, and the sum of their squares is $$9x^2 + 60y^2 = \square \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Put $y = 3z$, (1) becomes

$$x^2 + 60z^2 = \square = \left(x + \frac{p}{q}z\right)^2 \quad\ldots\ldots\ldots\ldots\ldots (2);$$

whence $$\frac{x}{z} = \frac{60q^2 - p^2}{2pq}.$$

Now, take $x = 60q^2 - p^2$, $z = 2pq$, and we have $y = 6pq$, and the required numbers are

$$60q^2 - 24pq - p^2, \quad 60q^2 - 18pq - p^2, \quad 60q^2 - 12pq - p^2,$$
$$60q^2 - 6pq - p^2, \quad 60q^2 - p^2, \quad 60q^2 + 6pq - p^2$$
$$60q^2 + 12pq - p^2, \quad 60q^2 + 18pq - p^2, \quad 60q^2 + 24pq - p^2;$$

also, the root of the sum of their squares is $3(60q^2 + p^2)$.

Take $p = 1$, $q = \frac{1}{3}$; then $x = 14$, $y = 3$, and the required numbers are $\quad 2, 5, 8, 11, 14, 17, 20, 23, 26$;

and we have $2^2 + 5^2 + 8^2 + 11^2 + 14^2 + 17^2 + 20^2 + 23^2 + 26^2 = 48^2$.

Take $p = 1$, $q = 1$; then $x = 59$, $y = 6$, and the numbers are

$$35, 41, 47, 53, 59, 65, 71, 77, 83;$$

and we have $35^2 + 41^2 + 47^2 + 53^2 + 59^2 + 65^2 + 71^2 + 77^2 + 83^2 = 183^2$.

By giving suitable values to p and q, an infinite number of sets may be found.

2. Take the well-known identity

$$(x+y)^2 = (x-y)^2 + 4xy \quad \text{.........................} \quad (3).$$

If we can transform $4xy$ into a square, we shall have two square numbers whose sum is a square. Since x may be any quantity whatever, we may put $\quad x = a+b+c+d+e+f+g+h$,

and then we have

$$(a+b+c+d+e+f+g+h+y)^2 = (a+b+c+d+e+f+g+h-y)^2$$
$$+ 4y(a+b+c+d+e+f+g+h) \quad \text{.........} \quad (4).$$

The last term will be a square if we take $a = i^2$, $b = j^2$, $c = k^2$, $d = l^2$, $e = m^2$, $f = n^2$, $g = p^2$, $h = q^2$, $y = r^2$; and we have

$$(i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2+r^2)^2$$
$$= (i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2-r^2)^2$$
$$+ (2ri)^2 + (2rj^2) + (2rk^2) + (2rl)^2 + (2rm)^2 + (2rn)^2 + (2rp)^2 + (2rq)^2 \quad \text{......} \quad (5).$$

Take $i = 1$, $j = 2$, $k = 3$, $l = 4$, $m = 5$, $n = 6$, $p = 7$, $q = 8$, $r = 9$; then, after dividing the numbers by 3, we have

$$6^2 + 12^2 + 18^2 + 24^2 + 30^2 + 36^2 + 41^2 + 42^2 + 48^2 = 95^2.$$

An infinite number of sets of nine square numbers whose sum is a square may be found from (5).

3463. (Professor Hudson, M.A.)—A solid of revolution possesses this property: A portion being cut off by a plane perpendicular to its axis and immersed vertex downwards in fluid, and then displaced through a small angle, the moment tending to restore equilibrium is independent of the amount cut off. Show that, if $y = f(x)$ be the generating curve, to determine f we have

$$[f(x)]^2 = \rho\left\{1 + [f'(x)]^2 + f(x)f''(x)\right\}\left\{f[x + f(x)f'(x)]\right\}^2,$$

ρ being the density of the solid compared with the fluid.

Solution by the Proposer.

Let k be the radius of the plane of flotation; h its distance from the vertex; k', h' corresponding quantities for the plane surface of the solid; and ρ the density of the solid compared with the fluid; then

$$\int_0^h \pi y^2\,dx = \rho\int_0^{h'} \pi y^2\,dx \quad \text{.........................} \quad (i.).$$

Also the moment tending to restore equilibrium varies as

$$\pi \frac{k^4}{4} + \int_0^h \pi y^2 x\, dx - \rho \int_0^{h'} \pi y^2 x\, dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)};$$

this is to be independent of h'; therefore, differentiating to h',

$$k^3 \frac{dk}{dh'} + k^2 h \frac{dh}{dh'} - \rho k'^2 h' = 0.$$

Also, from (1), $\quad k^2 \dfrac{dh}{dh'} = \rho k'^2$, whence $h' = h + k \dfrac{dk}{dh}$;

again, differentiating this to h,

$$\frac{dh'}{dh} = 1 + \left(\frac{dk}{dh}\right)^2 + k \frac{d^2k}{dh^2}; \quad \therefore \quad k^2 = \rho k'^2 \left\{ 1 + \left(\frac{dk}{dh}\right)^2 + k \frac{d^2k}{dh^2} \right\}.$$

Now, if $k = f(h)$, $\quad k' = f(h') = f\left(h + k \dfrac{dk}{dh}\right)$;

whence, writing x for h,

$$[f(x)]^2 = \rho \left\{1 + [f'(x)]^2 + f(x) f''(x)\right\} \left\{f[x + f(x) f'(x)]\right\}^2.$$

12690. (I. Arnold.)—On a given hypotenuse construct a right-angled triangle such that the product of the mth power of one leg and the nth power of the other leg may be a maximum.

Solution by D. Biddle; C. Bickerdike; *and others.*

Here we have $x < 1$; $x^m (1-x^2)^{\frac{1}{2}n}$ = a maximum. Obtaining the differential coefficient, we have

$$\left\{m(1-x^2) - nx^2\right\} x^{m-1} (1-x^2)^{\frac{1}{2}(n-2)} = 0,$$

whence $\qquad m(1-x^2) = nx^2$, and $x^2 = m/(m+n)$.

Therefore, on the given hypotenuse as base, draw a semicircle; and from the point in the base which divides it in the ratio $m : n$ draw a line at right angles. This line will cut the semicircle at the apex of the required triangle.

12713. (Professor Macfarlane.)—There are p electors and q candidates for r seats. Each elector has r votes, and he may distribute them as he pleases among the candidates. Find in how many different ways the voting may result, that is, the number of possible states of the poll.

Solution by H. W. Curjel, M.A.; Professor Sanjána; *and others.*

If we represent each candidate by a c and each vote by a v, each state of the poll will be represented by an arrangement of the pr like letters v and the q like letters c. Hence the question reduces to finding the

number of arrangements of $pr+q$ letters, pr being alike and q alike. If every voter uses all his votes, each arrangement must begin with a c; hence the number of states of the poll $= \dfrac{(pr+q-1)!}{(pr)!\ (q-1)!}$.
If each voter may use as few votes as he likes, the number of states of the poll $= \dfrac{(pr+q)!}{(pr)!\ q!}$, including the case when no one votes.

12485. (R. Knowles, B.A.)—PQ is a chord of an ellipse at right angles to the major axis; the diameter through Q meets the ellipse again in R; prove that PR′ drawn parallel to the tangent at R is the chord of curvature at P.

Solution by C. Bickerdike; Professor Droz-Farny; *and others.*

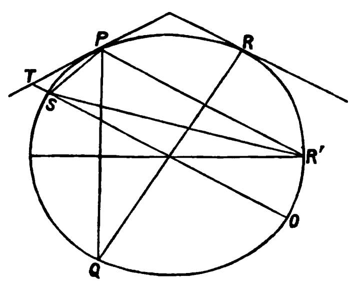
Because PR is parallel to the tangent at R, SO is parallel to PR.

Produce SO to cut the tangent at P in T, and draw PS and SR′. Then in the limit $\triangle$TPS is similar to the $\triangle$SPR′;

$\therefore$ PR : SP = SP : ST.

Hence PR = SP²/ST;

therefore PR′ is the chord of curvature at the point P.



12714. (Professor Scheffer.)—Suppose it to be possible to perform the passage through the North Pole: find (1) at what latitude would the maximum distance be saved by a ship sailing on the arc of a great circle instead of a parallel of latitude, the points of departure and destination being 180° apart; and (2) the maximum saving.

Solution by H. W. Curjel, M.A.; Professor Radhakushnan; *and others.*

Let θ be the co-latitude in circular measure; then, denoting the radius of the earth by unity, the distance saved by sailing on the arc of a great circle $= \pi \sin\theta - 2\theta$. This is a maximum when $\cos\theta = (2/\pi)$; therefore $\theta = \cos^{-1}(2/\pi)$, and the maximum saving $= (\pi^2-4)^{\frac{1}{2}} - 2\cos^{-1}(2/\pi)$.

12593. (H. Fortey.)—If

$$\phi(x, y, z, t, u) = \Sigma x^5 - 5\Sigma x^3(yu + zt) + 5\Sigma x(y^2u^2 + z^2t^2) - 5xyztu,$$

where in each case Σ means the sum of the term it precedes, and the four others derived from the same by cyclic permutation, show that

$$\{\phi(x, y, z, t, u)\}^2 = \phi(X, Y, Z, T, U),$$

where X, Y, Z, &c., are functions of x, y, z, &c., and determine these functions.

Solution by the Proposer.

1. Let α be any irrational fifth root of unity.

Then $\alpha^5 = 1$, $\alpha^6 = \alpha$, &c., also $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$, and in what follows, in every case where α would have an index greater than 4, the corresponding index not greater than 4 is substituted.

2. Write down the following five expressions :—

$$\left.\begin{array}{l} x + y + z + t + u \;\ldots\ldots 1 \\ x + \alpha y + \alpha^2 z + \alpha^3 t + \alpha^4 u \;\ldots\ldots 2 \\ x + \alpha^2 y + \alpha^4 z + \alpha t + \alpha^3 u \;\ldots\ldots 3 \\ x + \alpha^3 y + \alpha z + \alpha^4 t + \alpha^2 u \;\ldots\ldots 4 \\ x + \alpha^4 y + \alpha^3 z + \alpha^2 t + \alpha u \;\ldots\ldots 5 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Then, multiplying all these equations together, and remembering what is said in paragraph 1, their product will be found to be $\phi(x, y, z, t, u)$.

3. Next write down—

$$\left.\begin{array}{l} X + Y + Z + T + U \;\ldots\ldots 1' \\ X + \alpha^2 Y + \alpha^4 Z + \alpha T + \alpha^3 U \;\ldots\ldots 2' \\ X + \alpha^4 Y + \alpha^3 Z + \alpha^2 T + \alpha U \;\ldots\ldots 3' \\ X + \alpha Y + \alpha^2 Z + \alpha^3 T + \alpha^4 U \;\ldots\ldots 4' \\ X + \alpha^3 Y + \alpha Z + \alpha^4 T + \alpha^2 U \;\ldots\ldots 5' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Assume any expression in (b) to be the square in the corresponding expression in (a). Then, equating coefficients of the same powers of α, we have $\quad X = x^2 + 2yu + 2zt, \quad Y = y^2 + 2zx + 2tu,$ &c.,

where each capital, in terms of the small letters, follows the law of cyclic permutation. And we get the *same* values of X, Y, &c., whatever pair of expressions we take. But the expressions (b) are the same as (a), except that they are written in a different order, and capitals are substituted for small letters. Therefore their product is $\phi(X, Y, Z, T, U)$, and this has been shown to be equal to $\{\phi(x, y, z, t, u)\}^2$, when X, Y, &c., have the values given above.

12704. (Professor Droz-Farny. Suggested by Quest. 12613.)—Appliquer le théorème de Professeur Morel à la proposition bien connue angle BEA + CEA + DEA = 180°.

Solution by Rev. S. J. Rowton, M.A.; H. W. Curjel, M.A.; *and others.*

In the figure of 12613 (Vol. lxviii., p. 83),

$$\angle BEA + \angle CEA + \angle DEA = \angle GEH + \angle HED + \angle DEA = 180°.$$

12411. (Professor HUDSON, M.A.)—If the surface of a sphere is four times the area of its greatest section, or $\cdot 5236\ldots$ of the surface of the circumscribed cube, find what fraction the area of a circle is of the area of its circumscribed square.

Solution by R. CHARTRES, Professor BHATTACHARYA, *and others.*

By the Question. the area of a circle $= \frac{1}{4}(\cdot 5236\ldots) \times 6$ area of the square; therefore area of circle $= \cdot 7854$ area of circumscribed square.

12532. (R. TUCKER, M.A.)—

Prove that $\begin{vmatrix} k+a^2, & ab, & ac \\ ab, & k+b^2, & bc \\ ac, & bc, & k+c^2 \end{vmatrix}$ is divisible by k^3, where $k \equiv a^2+b^2+c^2$.

Solution by Professors DROZ-FARNY, BHATTACHARYA, *and others.*

En développant par la règle de SARRUS ce déterminant on trouve pour sa valeur $D \equiv k^3 + k^2(a^2+b^2+c^2)$ soit pour

$$k \equiv a^2+b^2+c^2, \quad D = 2k^3 = 2(a^2+b^2+c^2)^3.$$

12723. (G. E. CRAWFORD, M.A.)—Prove that the following series is convergent, and find its sum:—

$$\frac{2}{3.5}+\frac{2.4}{3.5.7}+\frac{2.4.6}{3.5.7.9}.$$

Solution by R. CHARTRES; H. W. CURJEL, M.A.; *and others.*

$$\int_0^{\frac{1}{2}\pi} \cos x \,.\log \sin x\, dx = (\sin x\,.\log\sin x - \sin x)_0^{\frac{1}{2}\pi} = -1;$$

but $\displaystyle\int_0^{\frac{1}{2}\pi} \cos x\,.\log\sin x\, dx = -\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos x \times \left\{\cos^2 x + \tfrac{1}{2}(\cos^4 x) + \tfrac{1}{3}(\cos^6 x) + \&c.\right\}$

$$= -\left\{\frac{1}{3}+\frac{2}{3.5}+\frac{2.4}{3.5.7}, \&c.\right\}; \quad \therefore \quad \frac{2}{3.5}+\frac{2.4}{3.5.7}+\&c. = \tfrac{2}{3}.$$

12687. (L. EAMONSON.) — Let nop (n) denote the number of odd primes occurring in $1, 2, 3, \ldots n$, and let nop(1) nop$(2n-1)$ + nop(3) nop $(2n-3)$ + nop(5) nop$(2n-5)$ + … + nop$(2n-1)$ nop$(1) = S_n$; prove that

the integer parts of $\frac{1}{2}(S_n - 2S_{n-1} + S_{n-2} + 1)$ = the number of partitions of $2n$ into two primes. [Ex.—The partitions of 6 into two primes are 1, 5 and 3, 3;

$$\text{nop}(1)\,\text{nop}(5) + \text{nop}(3)\,\text{nop}(3) + \text{nop}(5)\,\text{nop}(1) = 3+4+3 = 10 = S_3,$$
$$\text{nop}(1)\,\text{nop}(3) + \text{nop}(3)\,\text{nop}(1) = 4 = S_2,$$
$$\text{nop}(1)\,\text{nop}(1) = 1 = S_1, \quad \text{and} \quad \tfrac{1}{2}(10-8+1+1) = 2, \ \text{which is right.]}$$

Solution by Professors RADHAKUSHNAN, KRISHMACHANDRA DE, *and others.*

In the accompanying group, each row is a partition of $2n$ into two odd integers.

$$\begin{array}{cc} 1, & 2n-1 \\ 3, & 2n-3 \\ 5, & 2n-5 \\ 7, & 2n-7 \\ \vdots & \vdots \\ 2n-3, & 3 \\ 2n-1, & 1 \end{array}$$

And each of these is one of the partitions of the question, provided we omit the rows containing a non-prime number.

Now, as the rows equidistant from the extremes constitute the same partition, if m denote the number of rows (omitting the rows containing a non-prime number) in this group, then $\frac{1}{2}m$ if m be even, or $\frac{1}{2}(m+1)$ if m be odd, denotes the number of partitions of $2m$ into two primes; *i.e.*, the greatest integer in $\frac{1}{2}(m+1)$ in either case.

Any number $2n-1$ is evidently a prime number, if $\text{nop}(2n-1) - \text{nop}(2n-3)$ is not 0, but $= 1$. So the expression

$$U \equiv \text{nop}(1)\,\{\text{nop}(2n-1) - \text{nop}(2n-3)\}$$
$$+ \{\text{nop}(3) - \text{nop}(1)\}\,\{\text{nop}(2n-3) - \text{nop}(2n-5$$
$$+ \ldots + \{\text{nop}(2n-1) - \text{nop}(2n-3)\,\text{nop}(1)$$

contains a term $= 1$ for each of the rows constituting the required partition, and a term $= 0$ for each of the other rows.

Therefore the integer in $\frac{1}{2}(U+1)$ is the number of partitions of $2n$, and U is evidently $= (S_n - 2S_{n-1} + S_{n-2})$.

12695. (J. W. MULCASTER.)—An *able* writer obtains insertion for two articles out of every five sent in, and an *incapable* writer only obtains insertion for one out of every hundred sent in. Find the probability of a person being an *able* writer who obtains insertion for one in every seven sent in.

Solution by D. BIDDLE; Professor BHATTACHARYA; *and others.*

If x be the chance of a writer gaining insertion on any single occasion, then $x(1-x)^6$ represents his chance of gaining once and of failing six times out of seven. An able writer's chance ranges between the limits $\frac{2}{5}$ and 1; that of others, from 0 to $\frac{2}{5}$. The probability that the individual in question belongs to the latter class is given by

$$\int_0^{\frac{2}{5}} x(1-x)^6\,dx \bigg/ \int_0^1 x(1-x)^6\,dx.$$

Therefore

$$P = 1 - \int_0^{\frac{1}{2}} (x - 6x^2 + 15x^3 - 20x^4 + 15x^5 - 6x^6 + x^7)\,dx \Big/ \int_0^1 x(1-x)^6\,dx$$

$$= 1 - (\cdot 1863219 - \cdot 1703643)/(6\cdot 875 - 6\cdot 8571429)$$

$$= 1 - \tfrac{159576}{178571} = \tfrac{18995}{178571}, \text{ or a little over } \tfrac{2}{19}.$$

1087. (The Editor.)—ABCD is a conic whose centre is O. If the radii vectores OA, OB, OC, OD represent in magnitude and direction four forces, show that the direction of the resultant passes through the centre of a second conic which is parallel to the first, and passes through the points A, B, C, D.

Solution by H. J. Woodall, A.R.C.S.

Let A, B, C, D be (α), (β), (γ), (δ); then the resultant of OA, OB, OC, OD will be in magnitude and direction the join of $(0, 0)$ to $(a\Sigma\cos\alpha,\ b\Sigma\sin\alpha)$. A conic, with parallel axes, through A, B, C, D is given by

$$\begin{vmatrix} x^2/a^2, & y^2/b^2, & x/a, & y/b, & 1 \\ \cos^2\alpha, & \sin^2\alpha, & \dots & \dots & 1 \\ \cos^2\beta, & \dots & \dots & \dots & 1 \\ \cos^2\gamma, & \dots & \dots & \dots & 1 \\ \cos^2\delta, & \dots & \dots & \dots & 1 \end{vmatrix} = 0,$$

which reduces to
$$\begin{vmatrix} x^2/a^2 + y^2/b^2 - 1, & y^2/b^2 & \dots & 1 \\ 0 & & & \\ 0 & & & \end{vmatrix} = 0,$$

that is, to $x^2/a^2 + y^2/b^2 - 1 = 0$, which is the only conic.

12402. (J. J. Barniville, B.A.)—Prove that

$$1 + \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} - \dots = \tfrac{1}{5}\pi\sqrt{5}; \quad 1 + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \dots = \tfrac{1}{6}\pi\sqrt{6};$$

$$\frac{1}{1.2} - \frac{1}{6.7} + \frac{1}{11.12} - \dots = \tfrac{2}{5}\log 2 + \pi/\left\{5\,(5 + 2\sqrt{5})^{\frac{1}{2}}\right\}.$$

Solution by Rev. T. Roach, M.A.; Professor Sanjána; *and others*.

In the formula $\operatorname{cosec}\theta = \dfrac{1}{\theta} + \dfrac{1}{\pi - \theta} - \dfrac{1}{\pi + \theta} - \dfrac{1}{2\pi - \theta} + \dfrac{1}{2\pi + \theta} + \dots$, put $\theta = \frac{1}{10}\pi$, and multiply both sides by $\frac{1}{10}\pi$; thus

$$\tfrac{1}{10}\pi\,(\sqrt{5} + 1) = 1 + \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{19} + \tfrac{1}{21} \dots .$$

In the formula $\sec\theta = \dfrac{2}{\pi-2\theta}+\dfrac{2}{\pi+2\theta}-\dfrac{2}{3\pi-2\theta}-\dfrac{2}{3\pi+2\theta}+\dots,$ put $\theta=\frac{1}{5}\pi$, and multiply both sides by $\frac{1}{10}\pi$; thus

$$\tfrac{1}{10}\pi\,(\sqrt{5}-1) = \tfrac{1}{3}+\tfrac{1}{7}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{23}+\dots.$$

By addition the first result follows.

By substituting $\theta=\frac{1}{12}\pi$ in these formulæ, multiplying both sides by $\frac{1}{15}\pi$, and adding, the second result follows similarly.

$$1+\tfrac{1}{4}-\tfrac{1}{6}-\tfrac{1}{9}+\tfrac{1}{11}+\dots = \pi/(5\sin\tfrac{1}{5}\pi);\quad \tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{12}+\dots = \pi/(5\sin\tfrac{2}{5}\pi);$$

$$\therefore\quad 1-\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4},\quad -\tfrac{1}{6}+\dots = 2\pi/\left[5\sqrt{(5+2\sqrt{5})}\right].$$

Also
$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4},\quad -\tfrac{1}{6}+\dots = \tfrac{4}{5}\log 2;$$

$$\therefore\quad \frac{1}{1.2}-\frac{1}{6.7}+\frac{1}{11.12}-\dots = \pi/\left[5\sqrt{(5+2\sqrt{5})}\right]+\tfrac{2}{5}\log 2;$$

$$\frac{1}{3.4}-\frac{1}{8.9}+\frac{1}{13.14}-\dots = \pi/\left[5\,(5+2\sqrt{5})\right]-\tfrac{2}{5}\log 2.$$

12615. (Professor DROZ-FARNY.)—Deux circonférences fixes O et O′ sont données. Une troisième circonférence O″ est tangente à la première et coupe la seconde orthogonalement. Lieu du centre de O″ en supposant que chacune des deux circonférences O et O′ passe par le centre de l'autre, et dans le cas général.

Solution by H. W. CURJEL, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let a, b be the radii of O and O′, and c the distance between their centres, and x, y the coordinates of the centre of O″, the origin being at the centre of O and the axis of x along the line joining the centres of O and O′. Then

$$x^2-2cx+c^2+y^2 = b^2+\left\{(x^2+y^2)^{\frac{1}{2}}-a\right\}^2 = a^2+b^2+x^2+y^2-2a\,(x^2+y^2)^{\frac{1}{2}};$$

$$\therefore\ 2a\,(x^2+y^2)^{\frac{1}{2}} = 2cx+a^2+b^2-c^2;\qquad \therefore\ 4a^2\,(x^2+y^2) = (2cx+a^2+b^2-c^2)^2;$$

therefore locus is a conic.

When $a=b=c$, this becomes $4\,(x^2+y^2) = 4x^2+4ax+a^2$; *i.e.*, $4y^2 = 4ax+a^2$, a parabola.

9205. (Professor WOLSTENHOLME, M.A., Sc.D.) — Denoting by $\left\langle\begin{matrix}a, p\\ b, q\\ c, r\end{matrix}\right|$ the volume of a tetrahedron in which a, b, c are conterminous edges, and p, q, r the edges respectively opposite; prove that, if

$$\left\langle\begin{matrix}a, p\\ b, q\\ c, r\end{matrix}\right| = \left\langle\begin{matrix}a, p\\ b, r\\ c, q\end{matrix}\right| = \left\langle\begin{matrix}a, r\\ b, q\\ c, p\end{matrix}\right| \quad\dots\dots\dots\dots\dots\quad (1),$$

then will also $\left\langle\begin{matrix} a, p \\ b, q \\ r, c \end{matrix}\right| = \left\langle\begin{matrix} a, p \\ b, c \\ r, q \end{matrix}\right| = \left\langle\begin{matrix} a, c \\ b, q \\ r, p \end{matrix}\right|$ .... (2);

and that, in either of these systems, $A_1 + A_2 = 180°$, $B_1 + B_3 = 180°$, $B_2 = A_3$, the dihedral angles opposite to the edges a, b being denoted by A, B with a suffix 1, 2, or 3 corresponding to the tetrahedron.

[The lengths a, b, c, p, q, r are assumed to be all unequal. The equations

$$\left\langle\begin{matrix} a, p \\ b, c \\ q, r \end{matrix}\right| = \left\langle\begin{matrix} a, p \\ b, r \\ q, c \end{matrix}\right| \;\ldots\ldots\; (3), \qquad \left\langle\begin{matrix} a, c \\ b, q \\ p, r \end{matrix}\right| = \left\langle\begin{matrix} a, r \\ b, q \\ p, c \end{matrix}\right| \;\ldots\ldots\; (4),$$

also hold, and for these pairs the dihedral angles opposite a in the first pair, and opposite b in the second pair, are supplementary. The relations which must hold between the lengths of the edges are the two

$$a^2 + b^2 = 2(c^2 + r^2), \quad 3(a^2 - b^2) = 2(q^2 - p^2).]$$

Solution by H. J. Woodall, A.R.C.S.

Denote by V′ the common volume of the first three tetrahedra; then

$$\begin{aligned} 144V'^2 = & -p^2q^2r^2 + a^2p^2(q^2 + r^2 - p^2) + b^2q^2(r^2 + p^2 - q^2) + c^2r^2(p^2 + q^2 - r^2) \\ & -p^2(a^2 - b^2)(a^2 - c^2) - q^2(b^2 - c^2)(b^2 - a^2) - r^2(c^2 - a^2)(c^2 - b^2) = V'_1 \\ = & -p^2q^2r^2 + a^2p^2(q^2 + r^2 - p^2) + b^2r^2(q^2 + p^2 - r^2) + c^2q^2(p^2 + r^2 - q^2) \\ & -p^2(a^2 - b^2)(a^2 - c^2) - r^2(b^2 - c^2)(b^2 - a^2) - q^2(c^2 - a^2)(c^2 - b^2) = V'_2 \\ = & -p^2q^2r^2 + a^2r^2(q^2 + p^2 - r^2) + b^2q^2(p^2 + r^2 - q^2) + c^2p^2(r^2 + q^2 - p^2) \\ & -r^2(a^2 - b^2)(a^2 - c^2) - q^2(b^2 - c^2)(b^2 - a^2) - p^2(c^2 - a^2)(c^2 - b^2) = V'_3. \end{aligned}$$

From $V'_1 = V'_2$ and $V'_1 = V'_3$, we get

$$p^2 - q^2 - r^2 = b^2 + c^2 - 2a^2, \quad q^2 - p^2 - r^2 = a^2 + c^2 - 2b^2;$$

whence $\quad 2(p^2 - q^2) = 3(b^2 - a^2)$ and $2(c^2 + r^2 = a^2 + b^2$.

These relations prove the equality of tetrahedra in (2).

$$\sin A_1 = \frac{\text{perp. on } p \text{ from vertex of tet.}}{\text{perp. on base from vertex of tet.}} = \frac{2\,\text{area}\,(b\,.\,c\,.\,p)}{p} \Big/ \frac{3V_1'}{\text{area}\,(p\,.\,q\,.\,r)}$$
$$= \frac{2\,\text{area}\,(b\;c\,.\,p)\;\text{area}\,(p\,.\,q\,.\,r)}{3pV_1'} = \sin A_2;$$

therefore $A_1 = A_2$ or $A_1 + A_2 = 180°$. We cannot have $A_1 = A_2$, because that would involve $b = c$; hence $A_1 + A_2 = 180°$. Similarly, $B_1 + B_3 = 180°$.

Again, $\quad \sin B_2 = \{2\,\text{area}\,(a\,.\,c\,.\,r)\;\text{area}\,(p\,.\,q\,.\,r)\}/(3rV_2')$,

$\qquad\qquad \sin A_3 = \{2\,\text{area}\,(b\,.\,c\,.\,r)\;\text{area}\,(p\,.\,q\,.\,r)\}/(3rV_3')$,

$V_2' = V_3'$, and area $(a\,.\,c\,.\,r)$ = area $(b\,.\,c\,.\,r)$, easily proved since $a^2 + b^2 = 2(c^2 + r^2)$. Hence, $\sin B_2 = \sin A_3$.

We can, also, easily show the identities (3), (4).

[The relation $p^2 - q^2 - r^2 = b^2 + c^2 - 2a^2$ is suggestive, when written $b^2 + c^2 + q^2 + r^2 = p^2 + 2a^2$, as it plainly indicates that b, c, q, r may be interchanged without altering the volume of the tetrahedron.]

APPENDIX.

UNSOLVED QUESTIONS.

3728. (Rev. A. F. Torry, M.A.)—Taking the eccentricity of the earth's orbit round the sun to be $\frac{1}{60}$, and the year to consist of $365\frac{1}{4}$ days, determine the greatest and least angular velocities of the earth about the sun.

3745. (Rev. Dr. Booth, F.R.S.)—If a and β are the semi-axes of the evolute of an ellipse, show that there is but one length of flexible radius or string that will describe an ellipse, and that the end of any other length but $\dfrac{\beta^3}{\beta^2-a^2}$ will describe a curve of a higher order than the second.

3747. (Rev. A. F. Torry, M.A.)—A convex lens is held so that the distance between a bright point and its image is the least possible; two other lenses are then introduced, one half-way between the first lens and the luminous point, the other half-way between the first lens and the image of the point. If the position of the image remains unaltered, the sum of the focal lengths of the three lenses will be zero.

3750. (Professor Sir R. E. Ball.)—Determine a geometrical construction for the snail of a repeating clock.

3759. (J. F. Moulton, M.A.)—A ray of light is refracted through a prism in a principal plane. Show that, if the dispersion of two neighbouring colours be a minimum, $\dfrac{\sin(3\phi'-2i)}{\sin\phi'} = 1-\dfrac{2}{\mu^2}$.

3761. (W. Siverly.)—One end of a rod, whose length is equal to the diameter of a spherical shell, is passed through a hole in the shell and made to touch every point of the interior surface: find the surface described by the other end of the rod.

3767. (Editor.)—Two houses stand 750 yards apart on the side of a hill of uniform slope, and at the respective distances of 600 and 150 yards from a brook, which runs in a straight line along the foot of the hill. A man starts from the first house to go to the brook for water, which he is to carry to the second house. Supposing that he can only walk half as fast in going uphill with the water as he can in going downhill to the brook, find the path he must take, and the distance he will have to go, in order to perform his work in the shortest possible time.

3771. (H. McColl, B.A.)—In the quadratic equation $ax^2 - bx + c$, the numerical values of the coefficients a, b, c are each taken at random between 1 and 10. What is the chance that the equation has two real roots between 1 and 10?

3772. (Elizabeth Blackwood.)—In the equation $ad + (m + d)w = (m - d)x$, it is required to find the most probable value of x when d is found from observation, the limits of error in the observation being h and $-h$. The other quantities a and m are supposed to be known accurately.

3779. (Professor Hudson, M.A.) — There are n problems of equal difficulty upon a paper, for which na minutes are allowed. A man who could do any one of them in na minutes tries for a minutes at each. If the chance of his doing any one be proportional to the time he tries at it, what proportion of the marks for the paper may he expect to get?

3783. (Professor Sir R. E. Ball.)—If a rigid body can be rotated about three lines in space, then it can be screwed along the three axes of the hyperboloid containing those lines. Prove this, and show that the hitches of the turn-screws are inversely proportional to the squares of the axes. (N.B.—Small movement only.)

3784. (Editor.) — The width of a croquet hoop, the thickness of its wires, and the diameter of a ball, are given; the hoop being in a given position, show how to find the conditions that it may just be possible for it to go through the hoop (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection.

3785. (Myra Greaves.)—Two excursion trains, each m yards in length, may start with equal probability any time between 2 and 2·10 minutes from their respective stations in a direction at right angles to each other, each at a uniform rate v. Find the chances of a collision, each being n yards distant from the point at which their lines cross, and both being, of course, ignorant of the risk which they are running.

3789. (M. Collins, B.A.)—Has any one of the equations

$$x^3 - 7y^3 = 15\ldots,\quad x^3 - 29y^3 = -69\ldots,\quad x^3 - 29y^3 = -448\ldots$$

any solution in whole numbers besides $\dfrac{x}{y} = \dfrac{44}{23}, \dfrac{43}{14}, \dfrac{255}{83}$, respectively?

3793. (A. B. Evans, M.A.)—Find five square numbers such that the sum of every four of them shall be a square number.

3802. (J. Griffiths, M.A.)—A right circular cone rolls on a given plane round its vertex, which is fixed in the plane. Find the equations of the tortuous curve traced out by a point on the circumference of the base circle of the cone.

3813. (Professor Hudson, M.A.)—All vertical sections of a hill from the base to the summit are alike, and consist of two equal arcs of equal circles of which the lower has its convexity downwards and the upper has its convexity upwards, the highest and lowest tangents being horizontal: find whether a person who goes right over it or half round it traverses the greater distance. If the height of the hill be equal to the radius of either circle, find its apparent angular elevation from the base, and the height of equal towers at the base and summit, the tops of which are just mutually visible.

3827. (Professor Mukhopadhyay.)—One end of a rope is tied to a stake driven into the side of a square five-acre field, 12 yards from one corner, and the other is fastened to the hind leg of a cow, whose length is 6 feet, so as to allow the animal to graze over one-eighth part of the field. Required the length of the rope.

3828. (Professor Cayley, F.R.S.)—In a hexahedron ABCD, A′B′C′D′, the plane faces of which are ABCD, A′B′C′D′, A′ADD′, D′DCC′, C′CBB′, B′BAA′, the edges AA′, BB′, CC′, DD′ intersect in four points, say AA′, DD′ in α; BB′, CC′ in β; CC′, DD′ in γ; AA′, BB′ in δ: that is, starting with the duad of lines $\alpha\beta$, $\gamma\delta$, the four edges AA′, BB′, CC′, DD′ are the lines $\alpha\delta$, $\beta\delta$, $\beta\gamma$, $\alpha\gamma$, which join the extremities of these duads. Similarly, the four edges AB, CD, A′B′, C′D′ are the lines joining the extremities of a duad; and the four edges AD, BC, A′D′, B′C′ are the lines joining the extremities of a duad.

The question arises: "Given two duads, is it possible to place them in space so that the two tetrads of joining lines may be eight of the twelve edges of a hexahedron?"

[The duad $\alpha\beta$, $\gamma\delta$ is considered to be given when there is given the tetrahedron $\alpha\beta\gamma\delta$ which determines the relative position of the two finite lines $\alpha\beta$ and $\gamma\delta$.]

3840. (Dr. Artemas Martin.)—Three equal coins, radii r, are thrown horizontally at random, one at a time, into a circular box, radius R. Required the probability that only one of the coins rests on the bottom of the box. Consider the three cases, $R > 3r$, $R = 3r$, $R < 3r$.

3847. (S. Watson.)—A line is drawn at random so as to cut a given rectangle, within which two points are taken at random. Find the chance that the points lie on opposite sides of the line.

3854. (Professor Sir R. E. Ball.)—From any point perpendiculars are drawn to the generators of the surface $z(x^2+y^2)-2mxy=0$. Show that the feet of the perpendiculars lie upon a plane ellipse.

3857. (Professor Whitworth.)—Two curves touch one another, and both are on the same side of the common tangent. If, in the plane of the curves, this tangent revolves about the point of contact, or if it move parallel to itself, show that the prime ratio of the nascent chords in the former case is the duplicate of their prime ratio in the latter case.

3862. (J. J. Walker, M.A.)—If O_1, O_2, O_3 are the centres of circles described to the spherical triangle ABC; prove that

$$\frac{\cos O_2O_1O_3}{\sin\frac{1}{2}A}=\frac{\cos O_3O_2O_1}{\sin\frac{1}{2}B}=\frac{\cos O_1O_3O_2}{\sin\frac{1}{2}C}=\frac{1-\cos A-\cos B-\cos C}{4\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}.$$

3864. (J. W. L. Glaisher, M.A., F.R.S.)—Verify that

$$u=\int_{a^{\frac{1}{2}}x^{\frac{1}{2}}}^{\infty}e^{-z^n-(a^nx^n/z)}\,dz$$

is a particular integral of the differential equation

$$\frac{d^2u}{dx^2}-n^2a^nx^{n-2}u=\tfrac{1}{4}e^{-2a^{\frac{1}{2}n}x^{\frac{1}{2}}}a^{\frac{1}{2}}x^{-\frac{3}{2}},$$

which is a particular case of the extended Riccati's equation

$$\frac{d^2u}{dx^2}-x^qu=f(x).$$

3877. (Professor Tait, F.R.S.)—Show that, whatever functions of x be represented by y and z, we have always

$$\frac{\int yzdx}{\int ydx} > \epsilon^{\left(\int y \log z\, dx\right) / \left(\int ydx\right)},$$

all the integrals being taken between the same limits of x, and all the quantities involved being positive.

3878. (Professor Whitworth.)—Prove that the total number of signals which can be made with n different flags on s different masts is one less than the coefficient of x^n in the expansion of $n!\,\epsilon^x (1-x)^{-s}$.

3879. (W. S. B. Woolhouse, F.R.A.S.)—A solid and homogeneous hemisphere of cast iron, the radius of which is 12 inches, is placed with its vertex downwards upon a horizontal plane, and another, of the same description, radius 8 inches, is similarly placed upon the former, with its vertex in contact with the centre of the base. Determine their simultaneous oscillations, supposing sliding to be prevented by the friction of the surfaces. Also suppose a slight vertical impact to be given to the upper hemisphere, and find the periods of the vibrations which will be communicated by it to the lower hemisphere, as well as the amount of friction at both surfaces. Again, such motions may be given as will cause the two hemispheres to oscillate freely in corresponding and equal times, either in the same or opposite directions; find the times of these oscillations, and the quantity of friction in both cases.

3880. (Professor Salmon, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that $aU_1 + bU_2 + cU_3 + dU_4 + eU_5$ may be either a perfect square L^2 or the product of two factors MN. Prove that the lines L, M touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

3881. (Rev. W. Roberts, M.A.)—Find the value of the definite integral

$$\int_0^{\frac{1}{2}\pi} \frac{\log(\alpha + \beta \cos\phi)\, d\phi}{\alpha + \beta \cos^2\phi}.$$

3882. (Professor Townsend, F.R.S.)—The circumscribed and inscribed circles of a variable triangle, plane or spherical, being supposed both fixed, show that, throughout the deformation of the triangle, velocity of A : velocity of B : velocity of C = $\cot\frac{1}{2}A : \cot\frac{1}{2}B : \cot\frac{1}{2}C$, in either case; and hence that, angular velocity of a : angular velocity of b : angular velocity of $c = a : b : c$ for the plane, and $= \tan\frac{1}{2}a : \tan\frac{1}{2}b : \tan\frac{1}{2}c$ for the spherical triangle.

3883. (M. Collins, B.A.)—The harmonic mean between the segments of any chord of a conic section passing through its focus is constant and = the semi-parameter: required a demonstration true and general for all the three conics.

3896. (S. Watson.)—Three points are taken at random within a given triangle; find the chance that they will all lie on one side of some one line that can be drawn through the centroid of the triangle.

3897. (J. B. Sanders.)—(1) A given weight W is kept at rest on a circular arc by a weight P attached to a cord which passes over a point M in a vertical line through the centre of the circle. Required the position of W, supposing no friction at M.

(2) Instead of a circle as in (1), let the curve be a hyperbola with its transverse diameter vertical, the point M being at its centre.

3907. (Professor Sylvester, F.R.S.)—If, in the equation

$$x^5 + 5ex^4 + 10e^2x^3 + 10\epsilon^2x^2 + 5\epsilon x + 1 = 0,$$

$e\epsilon = m$, and m is greater than 1 or less than $\frac{1}{4}$, or (abstraction made of the case where $e = \epsilon$) m is equal to 1 or equal to $\frac{1}{4}$, only one of the roots of the equation will be real. But if m is intermediate between 1 and $\frac{1}{4}$, such values may be assigned to e and ϵ as will cause three of the roots to be real, in which case the necessary (but not sufficient) condition must be satisfied that $e\epsilon$ shall be of the form $m^{\frac{1}{2} \pm \rho}$, where $\rho < \frac{1}{10}$.

3933. (W. S. B. Woolhouse, F.R.A.S.)—Logarithmic tables always give the nearest figure, so that the error in the last digit is always comprised within the limits $\pm\frac{1}{2}$. When any number (n) of logarithms are taken promiscuously and added together, determine the probability that the error of the sum will be contained within the limits $\pm k$, k being any given value less than $\frac{1}{2}n$.

3934. (Professor Hudson, M.A.)—If the happiness which a person derives from his property increase with the property but at a diminishing rate, prove that, if a certain amount of property is to be divided among a certain number of persons, the greatest happiness will be secured by giving them equal shares. What will be the case if the happiness increase with the property (1) uniformly, (2) at an increasing rate?

3935. (Professor Crofton, F.R.S.)—A plane disk is kept in equilibrium by tangential stresses acting round its contour (supposed convex). Show that, in proceeding round the contour, the stress must change in sign at least 4 times. If the *intensity* of the stress is constant all round the contour, show that, if it change sign at 4 points only, those points must be the angles of a parallelogram. If it change sign at 6 points ABCDEF, the parallelograms, 3 of whose vertices are ABC, DEF, respectively, have their 4 vertices coincident.

3942. (Rev. W. Roberts, M.A.)—Let the base AB of a spherical triangle be given, the sides of which, AC, BC, are subject to the condition

$$\cos AC \sin^{\frac{1}{2}} AC - \cos BC \sin^{\frac{1}{2}} BC = (\sin AC - \sin BC)\,(k - \sin AC - \sin BC)^{\frac{1}{2}},$$

k being a constant quantity. Prove that the system of curves (locus of C) obtained by supposing k to vary, is cut orthogonally by a system which is found by writing in the above equation the angle BAC instead of AC, and the supplement of ABC instead of BC.

3943. (Professor Sir R. E. Ball.)—If a rigid body have four degrees of freedom, show that the body can be rotated about any line which intersects both of a certain pair of fixed lines.

3948. (Editor.) — ABCD is a rigid square at rest in its vertical plane, and supported by a fixed horizontal axis through its centre, about which it can move freely. A sphere of weight W falls from rest through

a feet, and strikes with its lowest point the square at A. At the moment of impact an exactly similar sphere, rolling on a smooth horizontal table in the same plane, with the velocity acquired by the falling sphere at A, strikes with its foremost point the square at C. Required the velocity with which the rolling sphere will return, taking into account the elasticity ϵ of the spheres.

3949. (Dr. C. Taylor.) — A polygon enveloping an ellipse has its vertices on fixed confocals. Through each vertex an ellipse is drawn, having its vertices at the adjacent points of contact. Show that there exists a linear relation between the areas of their minor auxiliary circles.

3950. (Dr. Artemas Martin.)—A plank a inches long and b inches wide is cut at random into two rectangular pieces, and one piece placed on the other. Find the chance that the top piece will not fall off, supposing the plank just as likely to be cut lengthwise as crosswise, and one piece just as likely to be put on top as the other.

3961. (Professor Clifford, F.R.S.)—In a polyhedron having n summits and only triangular faces ($\triangle$-faced n-acron, CAYLEY), let every plane which contains three summits, but is not a face, be called a diagonal plane; and let every line which contains two summits, but is not an edge, be called a diagonal line: then (a) there is a surface of class $n-4$ touching all the diagonal planes; (b) this surface contains all the diagonal lines; (c) the conditions of passing through the diagonal lines and touching the diagonal planes are just sufficient to determine the surface and no more; and (d) when the surface touches the plane at infinity, the volume of the polyhedron is zero.

3967. (Professor Burnside, M.A.)—Determine a property of the surface which renders the double integral

$$\iint \left\{ \mathrm{N}\left(\frac{1}{\mathrm{R}} + \frac{1}{\mathrm{R}_1}\right) - \frac{\mathrm{P}^2}{\mathrm{RR}_1} \right\} dx\, dy \text{ a maximum;}$$

where N is the normal, P the perpendicular from the origin on the tangent plane, and R, R_1 the principal radii of curvature.

3968. (S. Watson.)—A semi-ellipse is placed in a vertical position, and a heavy particle is just set in motion at the highest point of the curve, and descends down it of its own weight; required the point at which it will leave the curve, and its velocity at that point.

3969. (Professor Sir R. E. Ball.)—Find the axis of the couple which cannot disturb the equilibrium of a body having two degrees of freedom; and show that, if a body has more than two degrees of freedom, it cannot in general be in equilibrium under the action of a couple.

3975. (P. O'Cavanagh.)—If a double point and also a point of inflexion of a cubic be projected to infinity, what will be the nature of the new projected cubic?

3977. (Professor Crofton, F.R.S.)—If two points are taken at random within a given circle, find the chance that (1) the inner of the two, (2) the outer, shall lie within a given concentric circle. Extend this problem to cases where more than two points are taken, or to other cases of boundary than a concentric circle.

3980. (Professor Clifford, F.R.S.)—It is known that, if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that, if five lines be given, the five points so belonging to their five tetragrams lie on a circle [MIQUEL's Theorem, see *Diary* for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles which meet in a point; and, if $2n+1$ lines be given, they determine in this manner $2n+1$ points which lie on a circle.

3982. (Editor.)—Two coins are thrown at random into a circular box; show that the probability of their both resting on the same diameter of the bottom of the box is

$$\frac{1}{\pi}\left\{2(\alpha+\beta)+\sin 2(\alpha+\beta)\right\},$$

where 2α, 2β are the angles which the coins subtend at the centre of the bottom of the box when placed close to the side.

3983. (Dr. Artemas Martin.)—The bottom of a circular box is covered with an adhesive substance, and two straight rods, each equal in length to the radius of the box, are dropped horizontally into it at random; find the probability that the rods are crossed in the box.

3984. (Professor Minchin, M.A.)—If E is the complete elliptic function of the second kind, with modulus k, and if k' is the complementary modulus, prove that

$$\mathrm{E} = \frac{\pi k'^2}{2}\left\{1+\Sigma(2n+1)\left(\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}\,.\,k^n\right)^2\right\},$$

n assuming all values from 1 to ∞.

3987. (Colonel John H. Fry.)—On a level prairie, in a latitude 40° N., upon the last day of May, a horse turns towards his shadow at 6 o'clock in the morning and follows his shadow regularly at the rate of 4 miles an hour, till the clock strikes 1 p.m., when he stops. Find how far the horse then is from the place whence he started; and determine the curve he describes.

3990. (Professor Burnside, M.A.)—Trace the relation between the characteristics of a curve of the mth degree having the maximum number of double points, and the curve enveloped by the line

$$(a_0, a_1, a_2, \ldots a_m)(\theta, 1)^m = 0,$$

where $a_0, a_1, a_2, \ldots a_m$ are linear functions of the coordinates, and θ a variable parameter.

4004. (Professor Crofton, F.R.S.)—Two points are taken at random within a circle, and the one furthest from the centre is then effaced. Two more are taken in like manner, and the operation repeated an infinite number of times. Determine the law of the distribution of the points which remain. Likewise, if the *nearest* one had been effaced. Extend the question to three or more points.

4006. (W. S. B. Woolhouse, F.R.A S.)—Any given triangle may be orthogonally projected from an equilateral triangle, or it may be orthogonally projected into an equilateral triangle; determine by an easy geometrical construction the magnitude of the equilateral triangle in each case.

4033. (Rev. Dr. Booth, F.R.S.)— **Three confocal surfaces of the second order intersect in a common point Q the vertex of a cone which envelopes a fourth confocal surface; find the equation of this cone referred to the normals of the three surfaces, at the common point Q, as axes of coordinates.**

4056. (M. Gardiner.) — **Defining the area of a curvilinear plane figure as by polar coordinates in the integral calculus, prove the following general theorem:—If, at one extremity, a variable of constant length touch in every position a plane closed re-entering curve of any form consisting of m right and of n left hoops, the area of the figure described by the other extremity in the course of a complete revolution differs from that of the original figure by $(m-n)$ times the area of a circle whose radius is equal to the constant length of the line.**

4060. (Rev. W. H. Laverty, M.A.)—The lengths of the seasons being known, find the eccentricity of the ecliptic.

4065. (Professor Sylvester, F.R.S.)—If

$$x = \xi\eta^2+\eta\zeta^2+\zeta\xi^2-3\xi\eta\zeta, \quad y = \xi^2\eta+\eta^2\zeta+\zeta^2\xi-3\xi\eta\zeta,$$
$$z = \xi^{\frac{1}{3}}\eta^{\frac{1}{3}}\zeta^{\frac{1}{3}}(\xi^2+\eta^2+\zeta^2-\xi\eta-\eta\zeta-\zeta\xi);$$

prove (1) that

$$x^3+y^3+z^3 = (\xi\eta^2+\eta\zeta^2+\zeta\xi^2-3\xi\eta\zeta)(\xi^2\eta+\eta^2\zeta+\zeta^2\xi-3\xi\eta\zeta)(\xi^3+\eta^3+\zeta^3-3\xi\eta\zeta);$$

and hence prove (2) that, if u, v, w satisfy the equation

$$u^3+v^3+w^3 = \mathrm{M}uvw,$$

the same equation will be satisfied when for u, v, w we substitute

$$\mathrm{U} = u^3v^6+v^3w^6+w^3u^6-3u^3v^3w^3, \quad \mathrm{V} = u^6v^3+v^6w^3+w^6u^3-3u^3v^3w^3,$$
$$\mathrm{W} = \frac{1}{\mathrm{M}}\left\{u^9+v^9+w^9-3u^3v^3w^3\right\};$$

also (3) express *geometrically* the relation between u, v, w; U, V, W.

4078. (Professor Burnside, M.A.)—If the sides ad, bd, cd of a tetrahedron $abcd$ meet a surface of the second order in the points a, a_1; b, b_1; c, c_1, respectively, prove that the sections of the surface by the planes bca_1, cab_1, abc_1, abc are all touched by another plane section of the surface.

4080. (J. M. Greenwood.)—(1) A right cone, 40 feet high, radius of the base 2 feet, is covered with a delicate ribbon 1 inch wide. Suppose a pigeon takes hold of the ribbon at the vertex of the cone, and begins to unwind it, keeping it straight all the time: determine the distance the pigeon must fly. (2) Suppose a prolate spheroid, whose transverse axis is 40 feet and conjugate axis is 10 feet, is covered with ribbon as in the first case, and the transverse axis is at right angles to a plane upon which the spheroid rests: determine the distance the pigeon must fly to unwind the ribbon.

4089. (Rev. J. Blissard.)—Prove that

$$\frac{m}{1}\cdot\frac{1^m}{2^2}-\frac{m(m-1)}{1\,.\,2}\cdot\frac{2^m}{3^2}+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}\cdot\frac{3^m}{4^2}-\&\text{c.}$$
$$=\frac{(-1)^m}{m+1}\left\{\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m+1}\right)\right\}.$$

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